## BINOMIAL THEOREM

A Binomial is an algebraic expression of two terms which are connected by the operation '+' (or) '-'

**For example**, *x+siny, 3x²+2x, cosx+sin x* etc… are binomials.

**Binomial Theorem for positive integer:**

If n is a positive integer then

| | |
|---|---|
| $(x+a)^n = nC_0 x^n a^0 + nC_1 x^{n-1} a^1 + \ldots + nC_r x^{n-r} a^r + \ldots + nC_{n-1} x^1 a^{n-1} + nC_n x^0 a^n$ | ----(1) |

**Some Expansions**

a) If we put a = -a in the place of a in

$$(x-a)^n = nC_0 x^n (-a)^0 + nC_1 x^{n-1}(-a)^1 + \ldots + nC_r x^{n-r}(-a)^r + \ldots + nC_{n-1} x^1 (-a)^{n-1} + nC_n x^0 (-a)^n$$

$$(x-a)^n = nC_0 x^n a^0 - nC_1 x^{n-1} a^1 + \ldots + (-1)^r nC_r x^{n-r} a^r + \ldots + (-1)^r nC_{n-1} x^1 a^{n-1} + \ldots + (-1)^r nC_n x^0 a^n$$

b) Put x =1 and a = x in (1)

$$(1+x)^n = 1 + nC_1 x + nC_2 x^2 + \ldots + nC_r x^r + \ldots + nC_n x^n$$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 \ldots\ldots + x^n \qquad \text{----------(2)}$$

c) Put x = 1 and a = -x in (1)

$$(1-x)^n = 1 - nC_1 x + nC_2 x^2 - \ldots + (-1)^r nC_r x^r + \ldots + (-1)^n nC_n x^n$$

$$(1-x)^n = 1 - nx + \frac{n(n-1)}{2!}x^2 - \frac{n(n-1)(n-2)}{3!}x^3 \ldots\ldots + (-1)^n x^n \text{-----------(3)}$$

(d) Replacing n by – n in equation (2)

$$(1+x)^{-n} = 1 - nx + \frac{n(n+1)}{2!}x^2 - \frac{n(n+1)(n+2)}{3!}x^3 \ldots\ldots + (-1)^n x^n \qquad \text{---------(4)}$$

e) Replacing n by – n in equation (3)

$$(1-x)^{-n} = 1 + nx + \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+2)}{3!}x^3 \ldots\ldots + x^n \qquad \text{-----------(5)}$$

**Special Cases**

1. $(1+x)^{-1} = 1 - x + x^2 - x^3 \ldots\ldots$
2. $(1-x)^{-1} = 1 + x + x^2 + x^3 \ldots\ldots$
3. $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 \ldots\ldots$
4. $(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 \ldots\ldots$

**Note:**

1. There are *n+1* terms in the expansion of $(x+a)^n$.

2. In the expansion the general term is $nC_r\, x^{n-r} a^r$. Since this is the $(r+1)^{th}$ term, it is denoted by $T_{r+1}$ i.e. $T_{r+1} = nC_r\, x^{n-r} a^r$.

3. $nC_0, nC_1, nC_2, \ldots nC_r, \ldots nC_n$ are called binomial coefficients.

4. From the relation $nC_r = nC_{n-r}$, we see that the coefficients of terms equidistant from the beginning and the end are equal.

**Note:** The number of terms in the expansion of $(x+a)^n$ depends upon the index *n*. the index is either even (or) odd. Then the middle term is

**Case(i):** *n* is even

The number of terms in the expansion is *(n+1)*, which is odd.

Therefore, there is only one middle term and is given by $T_{\frac{n}{2}+1}$

**Case(ii) :** *n* is odd

The number of terms in the expansion is *(n+1),* which is even.

Therefore, there are two middle terms and they are given by $T_{\frac{n+1}{2}}$ and $T_{\frac{n+3}{2}}$

**Examples**

1. Expand (i) $\left(2x^2 + \frac{1}{x}\right)^6$

2. Find $11^7$.

**Solution:**

$11^7 = (1+10)^7$

$$= 7C_0(1)^7(10)^0 + 7C_1(1)^6(10)^1 + 7C_2(1)^5(10)^2 + 7C_3(1)^4(10)^3 + 7C_4(1)^3(10)^4 + 7C_5(1)^2(10)^5 + 7C_6(1)^1(10)^6 + 7C_7(1)^0(10)^7$$

$$= 1 + 70 + \frac{7\times6}{1\times2}10^2 + \frac{7\times6\times5}{1\times2\times3}10^3 + \frac{7\times6\times5\times4}{1\times2\times3\times4}10^4 + \frac{7\times6\times5\times4\times3}{1\times2\times3\times4\times5}10^5 + \frac{7\times6\times5\times4\times3\times2}{1\times2\times3\times4\times5\times6}10^6 + \frac{7\times6\times5\times4\times3\times2\times1}{1\times2\times3\times4\times5\times6\times7}10^7$$

= 1+ 70 + 2100 +35000 + 350000 + 2100000 + 7000000 + 10000000

= 19487171

2. Find the coefficient of $x^5$ in the expansion of $\left(x + \frac{1}{x^3}\right)^{17}$

**Solution**

In the expansion of $\left(x + \frac{1}{x^3}\right)^{17}$, the general term is

$$T_{r+1} = 17C_r\, x^{17-r}\left(\frac{1}{x^3}\right)^r$$

$$= 17C_r\, x^{17-4r}$$

Let $T_{r+1}$ be the term containing $x^5$

then, 17-4$r$ = 5 $\Rightarrow$ $r$ = 3

$$\therefore \quad T_{r+1} = T_{3+1}$$

$$= 17C_r\, x^{17-4(3)} = 680\, x^5$$

$\therefore$ coefficient of $x^5$ = 680.

3. Find the constant term in the expansion of $\left(\sqrt{x} - \frac{2}{x^2}\right)^{10}$

**Solution**

In the expansion of $\left(\sqrt{x} - \frac{2}{x^2}\right)^{10}$, the general term is

$$T_{r+1} = 10C_r\left(\sqrt{x}\right)^{10-r}\left(\frac{-2}{x^2}\right)^r$$

$$= 10C_r\, x^{\frac{10-r}{2}}\frac{(-2)^r}{x^{2r}} = 10C_r\,(-2)^r\, x^{\frac{10-r}{2}-2r}$$

$$= 10C_r\,(-2)^r\, x^{\frac{10-5r}{2}}$$

Let $T_{r+1}$ be the Constant term then,

$$\frac{10-5r}{2} = 0 \Rightarrow r = 2$$

$\therefore$ The constant term $= 10C_2\,(-2)^2\, x^{\frac{10-5(2)}{2}}$

$$= \frac{10\times 9}{1\times 2}\times 4\times x^0 = 180$$