

Differential Equations

Differential equation is an equation in which differential coefficients occur.

A differential equation is of two types

- (1) Ordinary differential equation
- (2) Partial differential equation

An ordinary differential equation is one which contains a single independent variable.

Example:

$$\frac{dy}{dx} = 2 \sin x \qquad \frac{dy}{dx} + 4y = e^x$$

A partial differential equation is one containing more than one independent variable.

Examples

1. $x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = z$
2. $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$

Here we deal with only ordinary differential equations.

Definitions

Order

The order of a differential equation is the order of the highest order derivative appearing in it.

$$\frac{dy}{dx} + 4y = e^x \qquad \text{Order 1}$$

$$\frac{d^2 y}{dx^2} + 4 \left(\frac{dy}{dx} \right)^2 - 3y = 0 \qquad \text{Order -2}$$

Degree

The degree of a differential equation is defined as the degree of highest ordered derivative occurring in it after removing the radical sign.

Example

Give the degree and order of the following differential equation.

1) $5(x+y) \frac{dy}{dx} + 3xy = x^2$ degree -1, order -1

$$2) \left(\frac{d^2 y}{dx^2} \right)^3 - 6 \frac{dy}{dx} + xy = 20 \quad \text{degree -3, order -2}$$

$$3) \sqrt{1 + \frac{d^2 y}{dx^2}} = 3 \frac{dy}{dx} + 1$$

Squaring on both sides

$$1 + \frac{d^2 y}{dx^2} = 9 \left(\frac{dy}{dx} \right)^2 + 6 \frac{dy}{dx} + 1$$

degree -1, order 2

$$4) \left(1 + \frac{dy}{dx} \right)^3 = a^2 \left(\frac{d^2 y}{dx^2} \right)^2$$

$$1 + 3 \frac{dy}{dx} + 3 \left(\frac{dy}{dx} \right)^2 + \left(\frac{dy}{dx} \right)^3 = a^2 \left(\frac{d^2 y}{dx^2} \right)^2$$

degree - 2, order - 2

Note

If the degree of the differential equations is one. It is called a linear differential equation.

Formation of differential equations

Given the solution of differential equation, we can form the corresponding differential equation. Suppose the solution contains one arbitrary constant then differentiate the solution once with respect to x and eliminating the arbitrary constant from the two equations. We get the required equation. Suppose the solution contains two arbitrary constant then differentiate the solution twice with respect to x and eliminating the arbitrary constant between the three equations.

Solution of differential equations

- (i) Variable separable method,
- (ii) Homogenous differential equation
- iii) Linear differential equation

Variable separable method

Consider a differential equation $\frac{dy}{dx} = f(x)$

Here we separate the variables in such a way that we take the terms containing variable x on one side and the terms containing variable y on the other side. Integrating we get the solution.

Note

The following formulae are useful in solving the differential equations

(i) $d(xy) = xdy + ydx$

(ii) $d\left(\frac{x}{y}\right) = \frac{ydx - xdy}{y^2}$

(iii) $d\left(\frac{y}{x}\right) = \frac{xdy - ydx}{x^2}$

Homogenous differential equation

Consider a differential equation of the form

$$\frac{dy}{dx} = \frac{f_1(x, y)}{f_2(x, y)} \longrightarrow (i)$$

where f_1 and f_2 are homogeneous functions of same degree in x and y .

Here put $y = vx$ $\left. \begin{array}{l} \\ \end{array} \right\} \longrightarrow (ii)$
 $\therefore \frac{dy}{dx} = v + x \frac{dv}{dx}$

Substitute equation(ii) in equation (i) it reduces to a differential equation in the variables v and x . Separating the variables and integrating we can find the solution.

Linear differential equation

A linear differential equation of the first order is of the form $\frac{dy}{dx} + py = Q$, Where

p and Q are functions of x only.

To solve this equation first we find the integrating factor given by

Integrating factor = I.F = $e^{\int p dx}$

Then the solution is given by

$$ye^{\int p dx} = \int Qe^{\int p dx} dx + c \text{ where } c \text{ is an arbitrary constant.}$$