Lesson 23

Half Range Sine and Cosine Series

In this chapter, we start discussion on even and odd function. As mentioned earlier if the function is odd or even then the Fourier series takes a rather simple form of containing sine or cosine terms only. Then we discuss a very important topic of developing a desired Fourier series (sine or cosine) of a function defined on a finite interval by extending the given function as odd or even function.

23.1 Even and Odd Functions

A function is said to be an even about the point a if $f(a-x) = f(a+x)$ for all x and odd about the point a if $f(a-x) = -f(a+x)$ for all x. Further, note the following properties of even and odd functions:

a) The product of two even or two odd functions is again an even function.

b) The product of and even function and an odd function is an odd function.

Using these properties we have the following results for the Fourier coefficients

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,\mathrm{d}x = \frac{2}{\pi}\begin{cases} \displaystyle\int_0^{\pi} f(x)\cos(nx)\,\mathrm{d}x, & \text{when } f \text{ is even function about } 0 \\ 0, & \text{when } f \text{ is odd function about } 0 \end{cases}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,\mathrm{d}x = \frac{2}{\pi}\begin{cases} 0, & \text{when } f \text{ is even function about } 0 \\ \displaystyle\int_0^{\pi} f(x)\sin(nx)\,\mathrm{d}x, & \text{when } f \text{ is odd function about } 0 \end{cases}$$

From these observation we have the following results

23.1.1 Proposition

Assume that f is a piecewise continuous function on $[-\pi, \pi]$. Then

a) *If f is an even function then the Fourier series takes the simple form*

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) \qquad \textit{with} \qquad a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)\,\mathrm{d}x, n = 0, 1, 2, \ldots.$$

Such a series is called a cosine series.

b) *If f is an odd function then the Fourier series of f has the form*

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin(nx) \qquad \textit{with} \qquad b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,\mathrm{d}x, n = 1, 2, \ldots.$$

Such a series is called a sine series.

23.2 Example Problems

23.2.1 Problem 1

Obtain the Fourier series to represent the function $f(x)$

$$f(x) = \begin{cases} x, & \textit{when } 0 \le x \le \pi \\ 2\pi - x, & \textit{when } \pi < x \le 2\pi \end{cases}$$

Solution: The given function is an even function about $x = \pi$ and therefore

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(nx)\,\mathrm{d}x = 0.$$

The coefficient a_0 will be calculated as

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,\mathrm{d}x = \frac{1}{\pi}\left[\int_0^{\pi} x\,\mathrm{d}x + \int_{pi}^{2\pi} (2\pi - x)\,\mathrm{d}x\right] = \frac{1}{\pi}\left[\frac{\pi^2}{2} + \frac{\pi^2}{2}\right] = \pi$$

The other coefficients a_n are given as

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(nx)\,\mathrm{d}x = \frac{1}{\pi}\left[\int_0^{\pi} x\cos(nx)\,\mathrm{d}x + \int_{pi}^{2\pi} (2\pi - x)\cos(nx)\,\mathrm{d}x\right]$$

It can be further simplified as

$$a_n = \frac{2}{n^2\pi}\left[(-1)^n - 1\right] = \begin{cases} 0, & \text{when } n \text{ is even} \\ \frac{4}{n^2\pi}, & \text{when } n \text{ is odd} \end{cases}$$

Therefore, the Fourier series is given by

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left[\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\right] \text{ where } 0 \leq x \leq 2\pi. \tag{23.1}$$

In this case as the function is continuous and f' is piecewise continuous, the series converges uniformly to $f(x)$ and we can write the equality (23.1).

23.2.2 Problem 2

Determine the Fourier Series of $f(x) = x^2$ on [-π, π] and hence find the value of the infinite series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^2}$ and $\sum_{n=1}^{\infty}\frac{1}{n^2}$.

Solution: The function $f(x) = x^2$ is even on the interval [-π, π] and therefore b_n=0 for all n. The coefficient a_0 is given as

$$a_0 = \frac{1}{\pi}\int_{\pi}^{\pi} x^2\,\mathrm{d}x = \frac{x^3}{3\pi}\Big|_{-\pi}^{\pi} = \frac{2\pi^2}{3}.$$

The other coefficients can be calculated by the general formula as

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\cos(nx)\,\mathrm{d}x = \frac{2}{\pi}\int_{0}^{\pi} x^2\cos(nx)\,\mathrm{d}x = \frac{2}{\pi}\left[x^2\frac{\sin(nx)}{n}\Big|_0^{\pi} - \frac{1}{n}\int_0^{\pi} 2x\sin(nx)\,\mathrm{d}x\right]$$

Again integrating by parts we obtain

$$a_n = \frac{4}{n\pi}\left[x\frac{\cos(nx)}{n}\Big|_0^{\pi} - \int_0^{\pi}\frac{\cos(nx)}{n}\,\mathrm{d}x\right] = \frac{4}{n\pi}\left[\frac{\pi(-1)^n}{n} - 0\right] = \frac{4(-1)^n}{n^2}$$

Therefore the Fourier series is given as

$$x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4(-1)^n}{n^2}\cos(nx) \quad \text{for} \quad x \in [-\pi, \pi]. \tag{23.2}$$

If we substitute $x = 0$ in the equation (23.2) we get

$$0 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4(-1)^n}{n^2} \Rightarrow \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}.$$

If we now substitute $x = \pi$ in the equation (23.2) we get

$$\pi^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4(-1)^{2n}}{n^2} \Rightarrow \frac{1}{4}\frac{2\pi^2}{3} = \sum_{n=1}^{\infty}\frac{1}{n^2} \Rightarrow \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}.$$

23.3 Half Range Series

Suppose that $f(x)$ is a function defined on $(0, l]$. Suppose we want to express $f(x)$ in the cosine or sine series. This can be done by extending $f(x)$ to be an even or an odd function on $[-l, l]$. Note that there exists an infinite number of ways to express the function in the interval $[-l, 0]$. Among all possible extension of f there are two, even and odd extensions, that lead to simple and useful series:

a) If we want to express $f(x)$ in cosine series then we extend $f(x)$ as an even function in the interval $[-l, l]$.

b) On the other hand, if we want to express $f(x)$ in sine series then we extend $f(x)$ as an odd function in $[-l, l]$.

We summarize the above discussion in the following proposition

23.3.1 Proposition

Let f be a piecewise continuous function defined on $[0, l]$. The series

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l} \qquad \textit{with} \qquad a_n = \frac{2}{l} \int_0^l f(x) \cos \frac{n\pi x}{l} \, \mathrm{d}x$$

is called half range cosine series of f. Similarly, the series

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \qquad \textit{with} \qquad b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} \, \mathrm{d}x$$

is called half range sine series of f.

Remark: *Note that we can develop a Fourier series of a function f defined in $[0, l]$ and it will, in general, contain all sine and cosine terms. This series, if converges, will represent a l-periodic function. The idea of half range Fourier series is entirely different where we extend the function f as per our desire to have sine or cosine series. The half range series of the function f will represent a $2l$-periodic function.*

23.4 Example Problems

23.4.1 Problem 1

Obtain the half range sine series for e^x *in* $0 < x < 1$.

Solution: Since we are developing sine series of f we need to compute b_n as

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\mathrm{d}x = 2\int_0^1 e^x \sin n\pi x = 2\left[e^x \sin n\pi x\big|_0^1 - n\pi\int_0^1 e^x\cos n\pi x\,dx\right]\mathrm{d}x$$
$$= 2\left[-n\pi\{e^x\cos n\pi x\big|_0^1 + n\pi\int_0^1 e^x\sin n\pi x\,\mathrm{d}x\}\right] = -2n\pi(e(-1)^n - 1) - n^2\pi^2 b_n$$

Taking second term on the right side to the left side and after simplification we get

$$b_n = \frac{2n\pi\left[1 - e(-1)^n\right]}{1 + n^2\pi^2}$$

Therefore, the sine series of f is given as

$$e^x = 2\pi\sum_{n=1}^{\infty}\frac{n\left[1 - e(-1)^n\right]}{1 + n^2\pi^2}\sin n\pi x \quad \text{for} \quad 0 < x < 1$$

23.4.2 Problem 2

Let $f(x) = \sin\frac{\pi x}{l}$ *on* $(0, l)$. *Find Fourier cosine series in the range* $0 < x < l$.

Solution: Sine we want to find cosine series of the function f we compute the coefficients a_n as

$$a_n = \frac{2}{l}\int_0^l \sin\frac{\pi x}{l}\cos\frac{n\pi x}{l}\,\mathrm{d}x = \frac{1}{l}\int_0^l\left[\sin\frac{(n+1)\pi x}{l} + \sin\frac{(1-n)\pi x}{l}\right]\mathrm{d}x$$

For $n \neq 1$ we can can compute the integrals to get

$$a_n = \frac{1}{l}\left[-\frac{\cos\frac{(n+1)\pi x}{l}}{\frac{(n+1)\pi}{l}} + \frac{\cos\frac{(1-n)\pi x}{l}}{\frac{(n-1)\pi}{l}}\right]_0^l = \frac{1}{\pi}\left[-\frac{(-1)^{n+1}}{n+1} + \frac{1}{n+1} + \frac{(-1)^{n-1}}{n-1} - \frac{1}{n-1}\right]$$

It can be further simplified as

$$a_n = \begin{cases} 0, & \text{when } n \text{ is odd} \\ -\frac{4}{\pi(n+1)(n-1)}, & \text{when } n \text{ is even} \end{cases}$$

The coefficient a_1 needs to calculated separately as

$$a_1 = \frac{1}{l}\int_0^l \sin\frac{2\pi x}{l}\,\mathrm{d}x = \frac{1}{l}\left[\cos\frac{2\pi x}{l}\,\frac{l}{2\pi}\right]_0^l = \frac{1}{2\pi}(1-1) = 0$$

The Fourier cosine series of f is given as

$$\sin\frac{\pi x}{l} = \frac{2}{\pi} - \frac{4}{\pi}\left[\frac{\cos\frac{2\pi x}{l}}{1\cdot 3} + \frac{\cos\frac{4\pi x}{l}}{3\cdot 5} + \frac{\cos\frac{6\pi x}{l}}{5\cdot 7} + \ldots\right]$$

23.4.3 Problem 3

Expand $f(x) = x,\ \ 0 < x < 2$ in a (i) sine series and (ii) cosine series.

Solution: (i) To get sine series we calculate b_n as

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{L}\,\mathrm{d}x = \frac{2}{2}\int_0^2 x\sin\frac{n\pi x}{2}\,\mathrm{d}x$$

Integrating by parts we obtain

$$b_n = \left[x\cos\frac{n\pi x}{2}\left(-\frac{2}{n\pi}\right)\right]_0^2 + \frac{2}{n\pi}\int_0^2 \cos\frac{n\pi x}{2}\,\mathrm{d}x = -\frac{4}{n\pi}\cos n\pi.$$

Then for $0 < x < 2$ we have the Fourier sine series

$$x = -\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos n\pi}{n}\sin\frac{n\pi x}{2} = \frac{4}{\pi}\left(\sin\frac{\pi x}{2} - \frac{1}{2}\sin\frac{2\pi x}{2} + \frac{1}{3}\sin\frac{3\pi x}{2} + \ldots\right).$$

(ii) Now we express $f(x) = x$ in cosine series. We need to calculate a_n for $n \neq 0$ as

$$a_n = \frac{2}{2}\int_0^2 x\cos\frac{n\pi x}{2}\,\mathrm{d}x = \left[x\sin\frac{n\pi x}{2}\left(\frac{2}{n\pi}\right)\right]_0^2 - \int_0^2 \sin\frac{n\pi x}{2}\left(\frac{2}{n\pi}\right)\,\mathrm{d}x$$

After simplifications we obtain

$$a_n = \frac{2}{n\pi}\left(\frac{2}{n\pi}\right)\left[\cos\frac{n\pi x}{2}\right]_0^2 = \frac{4}{n^2\pi^2}\left(\cos n\pi - 1\right) = \frac{4}{n^2\pi^2}\left[(-1)^n - 1\right]$$

The coefficient a_0 is given as

$$a_0 = \int_0^2 x\,\mathrm{d}x = 2$$

Then the Fourier sine series of $f(x) = x$ for $0 < x < 2$ is given as

$$x = 1 + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{\left[(-1)^n - 1\right]}{n^2}\cos\frac{n\pi x}{2} = 1 - \frac{8}{\pi^2}\left(\cos\frac{\pi x}{2} + \frac{1}{3^2}\cos\frac{3\pi x}{2} + \frac{1}{5^2}\cos\frac{5\pi x}{2} + \ldots\right).$$

It is interesting to note that the given function $f(x) = x, \ \ 0 < x < 2$ is represented by two entirely different series. One contains only sine terms while the other contains only cosine terms.

Note that we have used series equal to the given function because the series converges for each $x \in (0, 2)$ to the function value. It should also be pointed out that one can deduce sum of several series by putting different values of $x \in (0, 2)$ in the above sine and cosine series.

Suggested Readings

Davis, H.F. (1963). Fourier Series and Orthogonal Functions. Dover Publications, Inc. New York.

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