## Lesson 27

## Linear Differential Equation of First Order

In this lesson we shall learn linear differential equations of first order. Such equations are very often used in applications. Solution strategies of solving such equations will be discussed. Further a another special form of differential equation which can be reduced to linear differential equation of first order will be studied.

### 27.1 Linear Differential Equation

A first order differential equation is called linear if it can be written in the form

$$
\begin{equation*}
\frac{d y}{d x}+P(x) y=Q(x) \tag{27.1}
\end{equation*}
$$

where $P$ and $Q$ are constants or function of $x$ only.
A method of solving (27.1) relies on multiplying the equation by a function called integrating function so that the left hand side of the differential equation can be brought under a common derivative. Suppose $R(x)$ is an integrating factor of the (27.1). Multiplying the (27.1) by $R(x)$, we obtain

$$
\begin{equation*}
R(x) \frac{d y}{d x}+P(x) R(x) y=Q(x) R(x) \tag{27.2}
\end{equation*}
$$

Suppose, we wish that the L.H.S of (27.2) is the differential coefficient of some product. Clearly, the term $R(x) \frac{d y}{d x}$ can only be obtained by differentiating the product $R(x) y(x)$. In other words, we wish to have

$$
\begin{equation*}
R(x) \frac{d y}{d x}+P(x) R(x) y(x)=\frac{d}{d x}(R(x) y(x)) \tag{27.3}
\end{equation*}
$$

This implies

$$
R(x) \frac{d y}{d x}+P(x) R(x) y(x)=R(x) \frac{d y}{d x}+y(x) \frac{d R}{d x}
$$

On cancelling the first term on both the sides we obtain

$$
P(x) R(x) y(x)=y(x) \frac{d R}{d x} \quad \Rightarrow \quad \frac{d R}{R}=R d x
$$

Integrating the above equation, we get $\log R=\int P d x$. Note that the constant of integration is not important here because the integrating factor will be used to multiplying both the sides of the differential equation and therefore it will be cancelled. Thus, an integrating factor (I.F.) of the differential Equation (27.1) is

$$
\begin{equation*}
R=e^{\int P d x} \tag{27.4}
\end{equation*}
$$

The Equation (27.2) now reduces to

$$
\frac{d}{d x}(R y)=Q R
$$

By integrating above equation, we have

$$
R y=\int R Q d x+c
$$

or

$$
y e^{\int P d x}=\int Q e^{\int P d x} d x+c,
$$

which is required solution of given differential equation. Here $C$ is the constant of integration.

### 27.2 Example Problems

### 27.2.1 Problem 1

Solve $x \cos x \frac{d y}{d x}+y(x \sin x+\cos x)=1, \quad 0<x<\pi / 2$.
Solution: We rewrite the given equation as

$$
\frac{d y}{d x}+\left(\tan x+\frac{1}{x}\right)=\frac{\sec x}{x} .
$$

An I.F. of the given differential equation is

$$
e^{\int\left(\tan x+\frac{1}{x}\right) d x}=e^{\log x \sec x}=x \sec x .
$$

Hence, the required solution is

$$
y x \sec x=\int \sec ^{2} x d x+c
$$

or

$$
y x \sec x=\tan x+c,
$$

where, $c$ is an arbitrary constant.

### 27.2.2 Problem 2

Solve $\left(1+x^{2}\right) \frac{d y}{d x}=x(1-y)$.
Solution: Rewriting the given differential equation in standard form

$$
\frac{d y}{d x}+\frac{x}{1+x^{2}} y=\frac{x}{1+x^{2}}
$$

The I.F. is

$$
\text { I.F. }=e^{\int \frac{x}{1+x^{2}} d x}=e^{\frac{1}{2} \ln \left(1+x^{2}\right)}=\sqrt{1+x^{2}} .
$$

The solution is

$$
y \sqrt{1+x^{2}}=\int \frac{x}{\sqrt{1+x^{2}}}+c \Rightarrow y=1+c\left(1+x^{2}\right)^{-1 / 2}
$$

Here $c$ is an arbitrary constant.

### 27.3 Equations Reducible to Linear Form

A equation of the form

$$
\begin{equation*}
f^{\prime}(y) \frac{d y}{d x}+P f(y)=Q \tag{27.5}
\end{equation*}
$$

can be reduced to linear form, by substituting $f(y)=v$ so that $f^{\prime}(y) \frac{d y}{d x}=d v / d x$. The Equation (27.5) then becomes

$$
\begin{equation*}
d v / d x+P v=Q \tag{27.6}
\end{equation*}
$$

which is linear in $v$ and $x$ and its solution can be obtained with the help of I.F. as before. Thus, we have an I.F. $=e^{\int p d x}$ and the solution is

$$
v e^{\int p d x}=\int Q e^{\int P d x} d x+c
$$

Finally, we replace $v$ by $f(y)$ to obtain the required solution.

### 27.4 Example Problems

### 27.4.1 Problem 1

Solve $\frac{d y}{d x} \cos y+2 x \sin y=x$.

Solution: Substitution $\sin y=v$ which implies $\cos y \frac{d y}{d x}=\frac{d v}{d x}$ reduces the given differential equation to

$$
\frac{d v}{d x}+2 x v=x
$$

This is a linear differential equation of first order and its I.F. is $e^{\int 2 x d x}=e^{x^{2}}$. The solution of the equation in $v$ is given by

$$
v e^{x^{2}}=\int x e^{x^{2}} d x+c \Rightarrow v=\frac{1}{2}+c e^{-x^{2}}
$$

Replacing $v$ by $\sin y$ we get the required solution as

$$
y=\sin ^{-1}\left(\frac{1}{2}+c e^{-x^{2}}\right) .
$$

### 27.4.2 Problem 2

Solve $\frac{d y}{d x}+x \sin 2 y=x^{3} \cos ^{2} y$
Solution: Dividing the given differential equation by $\cos ^{2} y$, we obtain

$$
\sec ^{2} y \frac{d y}{d x}+2 x \tan y=x^{3}
$$

Putting $\tan y=v$ so that $\sec ^{2} y \frac{d y}{d x}=\frac{d v}{d x}$. Hence the above equation becomes

$$
\frac{d v}{d x}+2 x v=x^{3}
$$

which is linear. Its I.F. is $e^{x^{2}}$ and its solution is given as follows

$$
\begin{aligned}
& v e^{x^{2}}=\int e^{x^{2}} x^{3} d x+c \\
& v e^{x^{2}}=\frac{1}{2}\left(x^{2}-1\right) e^{x^{2}}+c .
\end{aligned}
$$

Replacing $v$ by $\tan y$ we obtain the required solution.

### 27.5 Bernoulli's Equation

An equation of the form

$$
\begin{equation*}
d y / d x+P y=Q y^{n} \tag{27.7}
\end{equation*}
$$

where $P$ and $Q$ are constants or function of $x$ only and $n$ is constant except 0 and 1 is called Bernoulli differential equation. This equation can easily be solved by multiplying both sides by $y^{-n}$ as

$$
\begin{equation*}
y^{-n} d y / d x+P y^{1-n}=Q \tag{27.8}
\end{equation*}
$$

Setting $y^{1-n}=v$, so that $y^{-n} \frac{d y}{d x}=\frac{1}{(1-n)} \frac{d v}{d x}$, the Equation (27.8) becomes

$$
d v / d x+P(1-n) v=Q(1-n)
$$

which is linear in $v$ and $x$. Its I.F. is $e^{\int P(1-n) d x}$ and hence the required solution is

$$
y^{1-n} e^{\int P(1-n) d x}=\int Q e^{\int P(1-n) d x} d x+c
$$

where $c$ is an arbitrary constant.

### 27.5.1 Example

Solve $x \frac{d y}{d x}+y=y^{2} \ln x$.

Solution: Rewrite the given equation

$$
\begin{equation*}
y^{-2} \frac{d y}{d x}+\frac{1}{x} y^{-1}=-x^{-1} \ln x \tag{27.9}
\end{equation*}
$$

Putting $y^{-1}=v$ so that $-y^{-2} \frac{d y}{d x}=\frac{d v}{d x}$. Then the Equation (27.9) gives

$$
\begin{equation*}
\frac{d v}{d x}-\frac{1}{x} v=x^{-1} \ln x \tag{27.10}
\end{equation*}
$$

The I.F. of the differential Equation (27.10) is $e^{-\int \frac{1}{x} d x}=\frac{1}{x}$, and hence the solution becomes

$$
v \frac{1}{x}=-\int x^{-2} \log x d x+c
$$

or by replacing $v$ by $y^{-1}$ we get

$$
y^{-1}=1+\ln x+c x
$$

where $c$ is an arbitrary constant.

## Suggested Readings

Boyce, W.E. and DiPrima, R.C. (2001). Elementary Differential Equations and Boundary Value Problems. Seventh Edition, John Willey \& Sons, Inc., New York.

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