Lesson 33

Method of Undetermined Coefficients

In the last lesson we have discussed operator method of finding particular integral. In this lesson we lean method of undetermined coefficients for finding particular integral of non-homogeneous differential equations. This method is relatively easier to apply once a possible form of a particular integral is known. This method is mainly applicable to linear differential equations with constant coefficients.

33.1 Method of Undetermined Coefficients

The method of undetermined coefficients requires that we make an initial assumption about the form of a particular solution of the differential equation, but with the coefficients left unspecified. We then substitute the assumed expression into the given differential equation and attempt to determine the coefficients so as to satisfy that differential equation. If we are successful, then we have found a particular solution of the differential equation. If we cannot determine the coefficients, then this means that there is no solution of the form that we assumed. In this case we may modify the initial assumption and try again.

The main advantage of the method of undetermined coefficients is that it is straightforward to execute once the assumption is made as to the form of the particular solution. Its major limitation is that it is useful primarily for equations for which we can easily write down the correct form of the particular solution in advance. This method is usually used only for problems in which the homogeneous equation has constant coefficients and the nonhomogeneous term is restricted to a relatively small class of functions. In particular, we consider only nonhomogeneous terms that consist of polynomials, exponential functions, sines, and cosines. Despite this limitation, the method of undetermined coefficients is useful for solving many problems that have important applications.

We shall demonstrate the method by taking a couple of different examples.

33.2 Example Problems

33.2.1 Problem 1

Solve the following initial value problem

$$y'' + 5y' + 6y = 2x + 1 \tag{33.1}$$

with the initial conditions $y(0) = 0$ *and* $y'(0) = \frac{1}{3}$.

Solution: First we solve the corresponding homogeneous equation. The characteristic equation is

$$m^2 + 5m + 6 \quad \Rightarrow \quad m = -2, -3.$$

Hence the complementary function is

$$\text{C.F.} = C_1 e^{-2x} + C_2 e^{-3x}.$$

To find particular integral, the trick is to somehow to guess one particular solution to Equation (33.1). Note that $2x + 1$ is a polynomial, and the left hand side of the equation will be a polynomial if we let y be a polynomial of the same degree. Let us try

$$y_p = Ax + B$$

We plug in to the differential equation to obtain

$$\begin{aligned} y_p'' + 5y_p' + 6y_p =& (Ax+B)'' + 5(Ax+B)' + 6(Ax+B) \\ =& 0 + 5A + 6Ax + 6B = 6Ax + (5A + 6B). \end{aligned}$$

So $6Ax + (5A + 6B) = 2x + 1$. Therefore,

$$A = \frac{1}{3} \text{ and } B = \frac{-1}{9}$$

That means

$$y_p = \frac{1}{3}x - \frac{1}{9} = \frac{3x - 1}{9}$$

Hence the general solution to (33.1) is

$$y = C_1 e^{-2x} + C_2 e^{-3x} + \frac{3x - 1}{9}.$$

The general solution must satisfy the given initial conditions. First find

$$y' = -2C_1e^{-2x} - 3C_2e^{-3x} + \frac{1}{3}$$

Then

$$0 = y(0) = C_1 + C_2 - \frac{1}{9}, \qquad \frac{1}{3} = y'(0) = -2C_1 - 3C_2 + \frac{1}{3}.$$

We solve to get $C_1 = 1/3$ and $C_2 = -2/9$. The particular solution we want is

$$y(x) = \frac{1}{3}e^{-2x} - \frac{2}{9}e^{-3x} + \frac{3x-1}{9} = \frac{3e^{-2x} - 2e^{-3x} + 3x - 1}{9}.$$

33.2.2 Problem 2

Find a particular solution of the differential equation

$$y'' + 2y' + 2y = \cos(2x).$$

Solution: We start by guessing the solution that includes some multiple of $\cos(2x)$. We may have to also add a multiple of $\sin(2x)$ to our guess since derivatives of cosine are sines. We try

$$y_p = A\cos(2x) + B\sin(2x).$$

We plug y_p into the equation and we get

$$-4A\cos(2x) - 4B\sin(2x) - 4A\sin(2x) + 4B\cos(2x) + 2A\cos(2x) + 2B\sin(2x) = \cos(2x).$$

The left hand side must equal to right hand side. We group terms and get $-4A+4B+2A = 1$ and $-4B - 4A + 2B = 0$. So $-2A + 4B = 1$ and $2A + B = 0$ and hence $A = \frac{-1}{10}$ and $B = \frac{1}{5}$. Hence a particular solution is

$$y_p = A\cos(2x) + B\sin(2x) = \frac{-\cos(2x) + 2\sin(2x)}{10}.$$

Remark 1: *If the right hand side contains exponentials we try exponentials. For example, for*

$$Ly = e^{3x},$$

we will try $y = Ae^{3x}$ as our guess and try to solve for A.

Remark 2: *If the right hand side is a multiple of sines, cosines, exponentials, and polynomials, we can use the product rule for differentiation to come up with a guess. We need to guess a form for y_p such that Ly_p is of the same form, and has all the terms needed to for the right hand side. For example,*

$$Ly = (1 + 3x^2)\, e^{-x} \cos(\pi x).$$

For this equation, we will guess

$$y_p = (A + Bx + Cx^2)\, e^{-x} \cos(\pi x) + (D + Ex + Fx^2)\, e^{-x} \sin(\pi x).$$

We will plug in and then hopefully get equations that we can solve for $A, B, C, D, E,$ and F.

Remark 3: *If the right hand side has several terms, such as*

$$Ly = e^{2x} + \cos x.$$

In this case we find u that solves $Lu = e^{2x}$ and v that solves $Lv = \cos x$ (that is, do each term separately). Then note that if $y = u + v$, then $Ly = e^{2x} + \cos x$. This is because L is linear; we have $Ly = L(u + v) = Lu + Lv = e^{2x} + \cos x$.

33.2.3 Problem 3

Find a particular solution of

$$y'' - 3y' - 4y = 3e^{2t} + 2\sin t - 8e^t \cos 2t.$$

Solution: By splitting up the right side of the given differential equation, we obtain the three differential equations

$$y'' - 3y' - 4y = 3e^{2t}, \qquad y'' - 3y' - 4y = 2\sin t, \qquad y'' - 3y' - 4y = 8e^t \cos 2t$$

Solutions of these three equations can be found with appropriate guess of the particular integral discussed above. Finally, a particular solution is their sum, namely,

$$Y(t) = \frac{1}{2}e^{2t} + \frac{3}{17}\cos t \frac{5}{17}\sin t + \frac{10}{13}e^t cos2t + \frac{2}{13}e^t \sin 2t.$$

The procedure illustrated in these examples enables us to solve a large class of problems in a reasonably efficient manner. However, there is one difficulty that sometimes occurs. It could be that our guess actually solves the associated homogeneous equation. The next example illustrates how it arises.

33.2.4 Problem 4

Solve the following differential equation

$$y'' - 9y = e^{3x}$$

Solution: In order to find a particular integral an intelligent guess would be $y = Ae^{3x}$, but if we plug this into the left hand side of the equation we get

$$y'' - 9y = 9Ae^{3x} - 9Ae^{3x} = 0 \neq e^{3x}.$$

There is no way we can choose A to make the left hand side be e^{3x} because our guess satisfies homogeneous equation. Note that the general solution of the homogeneous equation is

$$\text{C.F.} = C_1e^{-3x} + C_2e^{3x}$$

Thus our assumed particular solution is actually a solution of the corresponding homogeneous equation; consequently, it cannot possibly be a solution of the nonhomogeneous equation. To find a particular solution we must therefore consider functions of a somewhat different form. We modify our guess to $y = Axe^{3x}$ and notice there is no difficulty anymore. Note that $y' = Ae^{3x} + 3Axe^{3x}$ and $y'' = 6Ae^{3x} + 9Axe^{3x}$. So

$$y'' - 9y = 6Ae^{3x} + 9Axe^{3x} - 9Axe^{3x} = 6Ae^{3x}.$$

Thus $6Ae^{3x}$ is supposed to equal e^{3x}. Hence, $6A = 1$ and so $A = \frac{1}{6}$. We can now write the general solution as

$$y = y_c + y_p = C_1e^{-3x} + C_2e^{3x} + \frac{1}{6}\,xe^{3x}.$$

33.2.5 Problem 5

Find a particular solution of

$$y'' + 4y = 3\cos 2t$$

Solution: First we write its complimentary function

$$\text{C.F.} = c_1 \cos 2t + c_2 \sin 2t$$

As in earlier example, we guess

$$y_p = At \cos 2t + Bt \sin 2t$$

Then, upon calculating y_p' and Y_p'', substituting them into the given differential equation, we find that

$$4A \sin 2t + 4B \cos 2t = 3cos2t$$

Therefore $A = 0$ and $B = 3/4$, so a particular solution of the given differential equation is

$$y_p(t) = \frac{3}{4} t \sin 2t$$

Remark 4: *It is also possible that multiplying by x does not get rid of the problem we had faced in last two examples. For example,*

$$y'' - 6y' + 9y = e^{3x}.$$

The complementary solution is $y_c = C_1 e^{3x} + C_2 x e^{3x}$. Guessing $y = Ae^{3x}$ or $y = Axe^{3x}$ would not get us anywhere. In this case we will guess $y_p = Ax^2 e^{3x}$.

Suggested Readings

Boyce, W.E. and DiPrima, R.C. (2001). Elementary Differential Equations and Boundary Value Problems. Seventh Edition, John Willey & Sons, Inc., New York.

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