Lesson 42

Gradient and Directional Derivative

42.1 Gradient of a Scalar Field

Let $f(x, y, z)$ be a real valued function defining a scalar field. To define the gradient of a scalar field, we first introduce a vector operator called del operator denoted by ∇. We define the vector differential operator in two and three dimensions as

$$\nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} \quad \text{and} \quad \nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$$

The gradient of a scalar field $f(x, y, z)$, denoted by ∇f or grad (f) is defined as

$$\nabla \mathrm{f} = i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z}$$

Note that the del operator ∇ operates on a scalar field and produces a vector field.

42.1. 1 Example

Find the gradient of the following scalar fields

(i) $f(x, y) = y^2 - 4xy$ at $(1,2)$,

Solution

$$\nabla \mathrm{f(x, y)} = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y}\right)(y^2 - 4xy) = -4y\, i + (2y - 4x)j$$

42.1. 2 Example

$\boldsymbol{r} = xi + yj + zk$, $|\boldsymbol{r}| = r$ and $\hat{r} = \boldsymbol{r}/r$, then show that grad$\left(\frac{1}{r}\right) = -\hat{r}/r^2$.

Solution

Grad$\left(\frac{1}{r}\right) = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\left(\frac{1}{r}\right) = i\left(-\frac{1}{r^2}\frac{\partial r}{\partial x}\right) + j\left(-\frac{1}{r^2}\frac{\partial r}{\partial y}\right) + k\left(-\frac{1}{r^2}\frac{\partial r}{\partial z}\right) = -\frac{1}{r^2}\left(\frac{x}{r}\, i + \frac{y}{r}j + \frac{z}{r}k\right)$

$$= -\frac{1}{r^2}\left(\frac{\boldsymbol{r}}{r}\right) = -\frac{\hat{r}}{r^2}$$

where $\hat{r} = (xi + yj + zk)/r$

42.1. 3 Geometrical Representation of the Gradient

Let $f(P) = f(x,y,z)$ be a differentiable scalar field. Let $f(x,y,z) = k$ be a level surface and $P_0(x_0,y_0,z_0)$ be a point on it. There are infinite number of smooth curves on the surface passing through the point P_0. Each of these curves has a tangent at P_0. The totality of these tangent lines form a tangent plane to the surface at a point P_0. A vector normal to this plane at P_0 is called the normal vector to the surface at this point.

Consider now a smooth curve C on the surface passing through a point P on the surface. Let $x = x(t), y = y(t), z = z(t)$ be the parametric representation of the curve C. Any point P on C has the position vector $r(t) = x(t)i + y(t)j + z(t)k.$ Since the curve lies on the surface, we have

$$f(x(t), y(t), z(t)) = k$$

Then, $\frac{d}{dt}\, f(x(t), y(t), z(t)) = 0$

By chain rule, we have $\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} = 0$

or $\left(i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z}\right).\left(i\frac{dx}{dt} + j\frac{dy}{dt} + k\frac{dz}{dt}\right) = 0$

or $\nabla f . r'(t) = 0$

Let $\nabla f(P) \neq 0$ and $r'(t) \neq 0$. Now $r'(t)$ is a tangent to C at the point P and lies in the tangent plane to the surface at . Hence $\nabla f(P)$ is orthogonal to every tangent vector at P. Therefore, $\nabla f(P)$ is the vector normal to the surface $f(x,y,z) = k$ at the point P.

42.1. 4 Example

We will find a unit normal vector to the surface $xy^2 + 2yz = 8$ at the point $(3,-2,1)$.

Let $f(x,y,z) = xy^2 + 2yz = 8$ then

$$\frac{\partial f}{\partial x} = y^2, \frac{\partial f}{\partial y} = 2xy + 2z \text{ and } \frac{\partial f}{\partial z} = 2y$$

Therefore

$$\nabla \mathrm{f} = i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z} = y^2 i + (2xy + 2z)j + 2yk$$

At $(3,-2,1)$, we obtain the normal vector as $\nabla \mathrm{f}(3,-2,1) = 4\mathrm{i} - 10\mathrm{j} - 4\mathrm{k}$. The unit normal vector at $(3,-2,1)$ is given by

$$\frac{4i - 10j - 4k}{\sqrt{16+100+16}} = \frac{2i - 5j - 2k}{\sqrt{33}}.$$

42.1. 5 Example

Here we will find the angle between the two surfaces $xlog\, z = y^2 - 1$ and $x^2y = 2 - z$ at the given point (1,1,1).

First note that the angle between two surfaces at a common point is the angle between their normals at that point. Now we have

$$f_1(x, y, z) = xlog\, z - y^2 + 1 = 0, \Delta f_1(x, y, z) = (\log z)i - 2yj + (x/z)k$$

$$\Delta f_1(1,1,1) = -2j + k = n_1$$

$$f_2(x, y, z) = x^2y - 2 + z = 0,\ \Delta f_2(x, y, z) = 2xyi + x^2j + k$$

$$\Delta f_2(1,1,1) = 2i + j + k = n_2$$

Therefore $\cos\theta = \left|\frac{n_1.n_2}{|n_1||n_2|}\right| = \frac{1}{\sqrt{30}}$ or $\theta = cos^{-1}\left(\frac{1}{\sqrt{30}}\right)$.

42.1.6 Properties of Gradient

Let f and g be any two differentiable scalar fields. The gradient satidfies the following algebraic properties,

$$\Delta(f + g) = \Delta f + \Delta g$$

$$\Delta(c_1 f + c_2 g) = c_1\Delta f + c_2\Delta g, \text{ where } c_1, c_2 \text{ are arbitrary constants}$$

$$\Delta(fg) = f\Delta g + g\Delta f$$

$$\Delta\left(\frac{f}{g}\right) = \frac{g\Delta f - f\Delta g}{g^2}$$

42.2 Directional Derivative

Let $f(P) = f(x, y, z)$ be a differentiable scalar field.

Then $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$ denotes the rates of change of f in the direction of x, y and z axis, respectively.

If $f(x, y, z) = k$ is the level surface and P_0 is any point, then $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$ at $P_0(x_0, y_0, z_0)$ denote the slopes of the tangent lines in the directions of i, j, k respectively. It is natural to give the definition of derivative in any direction which we call as the directional derivative.

Let $\hat{b} = b_1 i + b_2 j + b_3 k$ be any unit vector. Let P_0 be any point $P_0: a = a_1 i + a_2 j + a_3 k$.

Then, the position vector of any point Q on the line passing through P_0 and in the direction of $\hat{b}$ is given by

$$r = a + t\hat{b} = (a_1 + tb_1)i + (a_2 + tb_2)j + (a_3 + tb_3)k = x(t)i + y(t)j + z(t)k$$

This is, the point $Q(a_1 + tb_1, a_2 + tb_2, a_3 + tb_3)$ is on this line. Now, the vector formthe point P_0 to Q is given by $t\hat{b}$. Since $|\hat{b}|$=1, the distance from P_0 to Q is t. Then

$$\frac{\partial f}{\partial t} = \lim_{t\to 0} \frac{f(Q)-f(P)}{t}$$

if it exists, is called the directional derivative of f at the point P_0 in the direction to $\hat{b}$.

Therefore $\frac{\partial}{\partial t}\, f(x(t), y(t), z(t))$ is rate of change of f with respect to the distance t.

We have

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$

where

$\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$ are evaluated at $t = 0$.

We write

$$\frac{\partial f}{\partial t} = \left(i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z}\right).\left(i\frac{dx}{dt} + j\frac{dy}{dt} + k\frac{dz}{dt}\right) = \nabla f.\frac{dr}{dt}$$

But $\frac{dr}{dt} = \hat{b}$(a unit vector). Therefore, the directional derivative of f in the direction of $\hat{b}$ in given by

Directional derivative $=\nabla f.\hat{b}$ = grad(f). $\hat{b}$,

which is denoted by $D_b(f)$. Note that $\hat{b}$ is a unit vector. If the direction is specified by a vector u, then $\hat{b} = u/|u|$.

42.2.1 Example

We will determine the directional derivative of $f(x, y, z) = xy^2 + 4xyz + z^2$ at the point (1,2,3) in the direction of $3i + 4j - 5k$.

Consider

$$\nabla f = (y^2 + 4yz)i + (2xy + 4xz)j + (4xy + 2z)k.$$

At the point (1,2,3), we have $\nabla f = 28i + 16j + 14k$. The unit vector in the given direction is $\hat{b} = (3i + 4j - 5k)/5\sqrt{2}$.

Therefore

$$D_b(1,2,3) = \frac{1}{5\sqrt{2}}(28i + 16j + 14k).(3i + 4j - 5k) = \frac{78}{5\sqrt{2}}$$

Suggested Readings

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