## Lesson 44

## Line Integral

### 44.1 Introduction

Let $C$ be a simple curve. Let the parametric representation of $C$ be written as

$$
\begin{equation*}
x=x(t), y=y(t), z=z(t), a \leq t \leq b \tag{44.1.1}
\end{equation*}
$$

Therefore, the position vector of appoint on the curve $C$ can be written as

$$
\begin{equation*}
r(t)=x(t) i+y(t) j+z(t) k, a \leq t \leq b \tag{44.1.2}
\end{equation*}
$$

### 44.2 Line Integral with Respect to Arc Length

Let $C$ be a simple smooth curve whose parametric representation is given as Eqs.(1) and (2). Let $f(x, y, z)$ be continuous on $C$. Then, we define the line integral $f$ of over $C$ with respect to the arc length $s$ by

$$
\int_{C} f(x, y, z) d s=\int_{a}^{b} f(x(t), y(t), z(t)) \sqrt{x^{\prime}(t)^{2}+y^{\prime}(t)^{2}+z^{\prime}(t)^{2}} d t
$$

since

$$
d s=\frac{d s}{d t} d t=\sqrt{\left(\frac{d x}{d t}\right)^{2}+\left(\frac{d y}{d t}\right)^{2}+\left(\frac{d z}{d t}\right)^{2}} d t
$$

### 44.2.1 Example

Evaluate $\int_{C}\left(x^{2}+y z\right) d s$, where $C$ is the curve defined by $x=4 y, z=3$ form $\left(2, \frac{1}{2}, 3\right)$ to $(4,1,3)$.

## Solution

Let $x=t$. Then, $y=t / 4$ and $z=3$. Therefore, the curve $C$ represented by $x=t, y=\frac{t}{4}, z=3,2 \leq t \leq 3$.

We have $d s=\sqrt{17} / 4$.
Hence $\int_{C}\left(x^{2}+y z\right) d s=\frac{\sqrt{17}}{4} \int_{2}^{4}\left(t^{2}+\frac{3}{4} t\right) d t=\frac{139 \sqrt{17}}{24}$.

### 44.2.2 Line Integral of Vector Fields

Let $C$ be a smooth curve whose parametric representation is given in Eqs. (44.1.1) and (44.1.2). Let

$$
v(x, y, z)=v_{1}(x, y, z) i+v_{2}(x, y, z) j+v_{3}(x, y, z) k
$$

be a vector field that is continuous on $C$. Then, the line integral of $v$ over $C$ is defined by

$$
\begin{align*}
\int_{C} v \cdot d r= & \int_{C} v_{1} d x+v_{2} d y+v_{3} d z \\
& =\int_{C} v(x(t), y(t), z(t)) \cdot \frac{d r}{d t} d t \tag{44.2.1}
\end{align*}
$$

If $v=v_{1}(x, y, z) i$, then Eq.(44.2.1) reduces to

$$
\int_{C} v \cdot d r=\int_{C} v_{1} d x=\int_{C} v_{1}(x(t), y(t), z(t)) \frac{d x}{d t} d t
$$

Similarly, if $v=v_{2}(x, y, z) j$ or $v=v_{3}(x, y, z) k$, we respectively obtained

$$
\int_{C} v \cdot d r=\int_{C} v_{2} d x=\int_{C} v_{2}(x(t), y(t), z(t)) \frac{d y}{d t} d t
$$

and

$$
\int_{C} v \cdot d r=\int_{C} v_{3} d x=\int_{C} v_{3}(x(t), y(t), z(t)) \frac{d y}{d t} d t
$$

### 44.2.2 Example

Evaluate the line integral of $v=x y i+y^{2} j+e^{z} k$ over the curve $C$ whose parametric representation is given by $x=t^{2}, y=2 t, 0 \leq t \leq 1$.

## Solution:

The position vector of any point on $C$ is given by $r=t^{2} i+2 t j+t k$. We have

$$
\begin{aligned}
\int_{C} v \cdot \frac{d r}{d t} d t= & \int_{0}^{1}\left(2 t^{3} i+4 t^{2} j+e^{t} k\right) \cdot(2 t i+2 j+k) d t \\
& =\int_{0}^{1}\left(4 t^{4}+8 t^{2}+e^{t}\right) d t=\frac{37}{15}+e
\end{aligned}
$$

### 44.2.3 Example

Evaluate the integral $\int_{c}\left(x^{2}+y z\right) d z$, where $C$ is given by $x=t, y=t^{2}, z=3 t, 1 \leq t \leq 2$.

## Solution:

We have $\int_{c}\left(x^{2}+y z\right) d z=2 \int_{1}^{2}\left(t^{2}+3 t^{3}\right) d t=\frac{163}{4}$

### 44.3 Line Integral of Scalar Fields

Let $C$ be a smooth curve whose parametric representation is as given in Eqs. (44.1.1) and (44.1.2). Let $f(x, y, z), g(x, y, z)$ and $h(x, y, z)$ be scalar fields which are continuous at point over $C$. Then, we define a line integral as

$$
\int_{C} f(x, y, z) d x+g(x, y, z) d y+h(x, y, z) d z
$$

$$
=\int_{C}\left[f(x(t), y(t), z(t)) \frac{d x}{d t}+g(x(t), y(t), z(t)) \frac{d y}{d t}+h(x(t), y(t), z(t)) \frac{d z}{d t}\right] d t
$$

If $C$ is closed curve, then we usually write

$$
\int_{C} v \cdot d r=\oint_{C} v \cdot d r
$$

### 44.3.1 Example

Evaluate $\int_{C}(x+y) d x-x^{2} d y+(y+z) d z$, where $C$ is $x^{2}=4 y, z=x, 0 \leq t \leq 2$.

## Solution

First we consider parametric form of $C$ as $x=t, y=\frac{t^{2}}{4}, z=2,0 \leq t \leq 2$.
Therefore,

$$
\int_{C}(x+y) d x-x^{2} d y+(y+z) d z=\int_{0}^{2}\left[\left(t+\frac{t^{2}}{4}\right)-t^{2}\left(\frac{t}{2}\right)+\left(\frac{t^{2}}{4}+t\right)\right] d t=\frac{10}{3}
$$

### 44.4 Application of Line Integrals

In this section, we consider some physical applications of the concept of line integral.

### 44.4.1 Work Done By A Force

Let $v(x, y, z)=v_{1}(x, y, z) i+v_{2}(x, y, z) j+v_{3}(x, y, z) k$ be a vector function defined and continuous at every point on $C$. Then the line integral of tangential component of $v$ along the curve $C$ from a point $P$ to the point $Q$ is given by

$$
\int_{P}^{Q} v \cdot d r=\int_{C} v \cdot d r=\int_{c} v_{1} d x+v_{2} d y+v_{3} d z
$$

Let now $v=F$, a variable force acting on a particle which moves along a curve $C$. Then, the work $W$ done by the force $F$ in displacing the particle from the point $P$ to the point $P$ along the curve $C$ is given by

$$
W=\int_{P}^{Q} F . d r=\int_{C^{*}} F \cdot d r
$$

where $C^{*}$ is the part of $C$, whose initial and terminal point are $P$ and $Q$.
Suppose that $F$ is a conservative vector field. Then $F$ can be written as $F=\operatorname{grad}(f)$, where $f$ is a scalar potential(field). Then, the work done

$$
W=\int_{C^{*}} F \cdot d r=\int_{C^{*}} \operatorname{grad}(f) \cdot d r
$$

$$
=\int_{C^{*}}\left(\frac{\partial f}{\partial x} d x+\frac{\partial f}{\partial y} d y+\frac{\partial f}{\partial z} d z\right)=\int_{P}^{Q} d f=[f(x, y, z)]_{P}^{Q}
$$

### 44.4.1 Example

Find the work done by the force $F=-x y i+y^{2} j+z k$ in moving a particle over the circular path $x^{2}+y^{2}=4, z=0$ form ( $2,0,0$ ) to $(0,2,0)$.

## Solution

The parametric representation of the given curve is $x=2 \cot t, y=2 \sin t, z=0,0 \leq t \leq$ $\pi 2$. Therefore, work done $W$ is given by

$$
W=\int_{C} F \cdot d r=\int_{C}-x y d x+y^{2} d y+z d z
$$

$$
\int_{0}^{\pi / 2}\left[-4 \sin t \cos t(-2 \sin t)+4 \sin ^{2} t(2 \cos )\right] d t=\frac{16}{13}
$$

### 44.4.2 Circulation

A line integral of a vector field $v$ around a simple closed curve $C$ is defined as the circulation of $v$ around $C$.

Circulation $=\oint_{C} v . d r=\oint_{C} \quad v \cdot \frac{d r}{d s} d s=\oint_{c} \quad v . T d s$,
where $T$ is the tangent vector to $C$. For example, in fluid mechanics, let $v$ represents the velocity field of a fluid and $C$ be a closed curve in its domain. Then, circulation gives the amount by which the fluid tends to turn the curve rotating or circulating around $C$. If $\oint_{c}$ v.Tds $>0$ then the fluid tends to rotate $C$ in the anti-clockwise direction, while if $\oint_{c} v . T d s<0$, then the fluid tends to rotate $C$ in the clockwise direction perpendicular to $T$ at every point on $C$, then $\oint_{c} v . T d s=0$, that is the curve does not move at all.

### 44.5 Line Integral Independent of the Path

Let $\phi(x, y, z)$ be a differentiable scalar function. The differential of $\phi(x, y, z)$ is defined as

$$
d \phi=\frac{\partial \phi}{\partial x} d x+\frac{\partial \phi}{\partial y} d y+\frac{\partial \phi}{\partial z} d z=\operatorname{grad} \phi \cdot d r
$$

Therefore, a differential expression expre $d \phi=f(x, y, z) d x+g(x, y, z) d y+h(x, y, z) d z$ is an exact differential, if there exists a scalar function $\phi(x, y, z)$ such that

$$
d \phi=f(x, y, z) d x+g(x, y, z) d y+h(x, y, z) d z
$$

We now present the result on the independence of the path of a line integral

### 44.5.1 Theorem

Let $C$ be a curve in simply connected domain $D$ in space. Let $f, g$ and $h$ be continuous function having continuous first partial derivatives in $D$. Then $\int_{C} f d x+g d y+h d z$ is independent of path $C$ if and only if the integrand is exact differential in $D$.

### 44.5.2 Example

Show that $\int_{C} \frac{x d x+y d y}{\sqrt{x^{2}+y^{2}}}$ is independent of path of integration which does not pass through the origin. Find the value of the integral from the point $P(-1,2)$ to the point $Q(2,3)$.

## Solution

We have $f(x, y)=\frac{x}{\sqrt{x^{2}+y^{2}}}$ and $g(x, y)=\frac{y}{\sqrt{x^{2}+y^{2}}}$
Now $\frac{\partial f}{\partial x}=-x y /\left(x^{2}+y^{2}\right)^{3 / 2}$ and $\frac{\partial g}{\partial x}=-x y /\left(x^{2}+y^{2}\right)^{3 / 2}$
Since $\frac{\partial f}{\partial x}=\frac{\partial f}{\partial x}$, the integral is independent of any path of integration which does not pass through the origin. Also, the integrand is an exact differential. Therefore, there exists a function $\phi(x, y)$ such that

$$
\frac{\partial \phi}{\partial x}=f(x, y)=\frac{x}{\sqrt{x^{2}+y^{2}}} \text { and } \frac{\partial \phi}{\partial y}=g(x, y)=\frac{y}{\sqrt{x^{2}+y^{2}}}
$$

Integrating the first equation with respect to $x$, we get $\phi(x, y)=\sqrt{x^{2}+y^{2}}+h(y)$.
Substituting in $\frac{\partial \phi}{\partial y}=\frac{y}{\sqrt{x^{2}+y^{2}}}=\frac{y}{\sqrt{x^{2}+y^{2}}}+\frac{d h}{d y}$ or $\frac{d h}{d y}=0$ or $h(y)=k$, constant.
Hence $\phi(x, y)=\sqrt{x^{2}+y^{2}}+k$
Therefore, $\int_{C} \frac{x d x+y d y}{\sqrt{x^{2}+y^{2}}}=\int_{(-1,2)}^{(2,3)} d\left(\sqrt{x^{2}+y^{2}}\right)=\left[\sqrt{x^{2}+y^{2}}\right]_{(-1,2)}^{(2,3)}=\sqrt{13}-\sqrt{5}$

## Suggested Readings

Courant, R. and John, F. (1989), Introduction to Calculus and Analysis, Vol. II, SpringerVerlag, New York.

Jain, R.K. and Iyengar, S.R.K. (2002) Advanced Engineering Mathematics, Narosa Publishing House, New Delhi.

Jordan, D.W. and Smith, P. (2002) Mathematical Techniques, Oxford University Press, Oxford.

Kreyszig, E. (1999) Advanced Engineering Mathematics, John Wiley, New York.

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Wylie, C. R. and Barrett, L.C. (2003) Advanced Engineering Mathematics, Tata McGrawHill, New Delhi.

