Lesson 10

Quadratic Forms

10.1 Introduction

The study of quadratic forms began with the pioneering work of Witt. Quadratic forms are basically homogeneous polynomials of degree 2. They have wide application in science and engineering.

10.2 Quadratic Forms and Matrices

Let $A = (a_{ij})$ be a real square matrix of size n and x be a column vector $x = (x_1, x_2, \ldots, x_n)^T$. A *quadratic form* on n variables is an expression $Q = x^T A x$.

In other words,

$$Q = x^T A x = \left(x_1,\ x_2,\ \ldots,\ x_n\right)\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}.$$

$$= a_{11}x_1^2 + a_{12}x_1x_2 + \ldots + a_{1n}x_1x_n + a_{21}x_2x_1 + a_{22}x_2^2 + \ldots + a_{2n}x_2x_n + \ldots + a_{n1}x_nx_1 + a_{n2}x_nx_2 + \ldots + a_{nn}x_n^2 = \sum_{j=1}^{n}\sum_{i=1}^{n} a_{ij}x_ix_j\ .$$

The matrix A is called the *matrix of the quadratic form* Q. This matrix A need not be symmetric. However, in the following theorem we show that every quadratic form corresponds to a unique symmetric matrix. Hence there is one to one correspondence between symmetric matrices of size n and quadratic forms on n variables.

Theorem 10.2.1: For every quadratic form Q there is a unique symmetric matrix B such that $Q = x^T B\, x$.

Proof: We consider an arbitrary quadratic form $Q = x^T Ax$, with $A = (a_{ij})$. We construct a matrix $B = (b_{ij})$, where $b_{ij} = \dfrac{a_{ij} + a_{ji}}{2}$. This matrix B is symmetric and $x^T A\, x = x^T B\, x$, i.e. quadratic forms associated with A and B are the same.

Example 10.2.1: For $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$, the quadratic form associated with A is

$$Q = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

$= x_1^2 + 2x_1x_2 + 3x_1x_3 + 4x_2x_1 + 5x_2^2 + 6x_2x_3 + 7x_3x_1 + 8x_3x_2 + 9x_3^2.$

$= x_1^2 + 6x_1x_2 + 10x_1x_3 + x_2^2 + 14x_2x_3 + 7x_3x_1 + 9x_3^2.$

This quadratic form is equal to the quadratic form $x^T B\, x$ where $B = \begin{pmatrix} 5 & 3 & 5 \\ 3 & 5 & 7 \\ 5 & 7 & 9 \end{pmatrix}$ which is a symmetric matrix.

If D is a diagonal matrix then the quadratic form associated with D is called a *diagonal quadratic form*, that is if $D = \mathrm{diag}\,(a_{11}, a_{22}, \ldots, a_{nn})$ then $x^T D\, x = a_{11}x_1^2 + a_{22}x_2^2 + \ldots + a_{nn}x_n^2$. This is also called the *canonical representation* of a quadratic form. The theorem below says that every quadratic form has a canonical representation.

Theorem 10.2.2: every quadratic form $x^T A x$ can be reduced to a diagonal quadratic form $y^T D y$ through a non-singular transformation $Px = y$, that is, P is a non-singular matrix.

The above theorem says that, for $x = (x_1, x_2, \ldots, x_n)^T$ and $y = (y_1, y_2, \ldots, y_n)^T$, variables $x_1, x_2, \ldots, x_n$ in $x^T A x$ can be changed to $y_1, y_2, \ldots, y_n$ through $Px = y$, P is a non-singular matrix, so that $x^T A x = y^T D y$, where D is a diagonal matrix.

We shall explain the above result through some examples.

Example 10.2.2: We reduce the quadratic forms **(a)** $4x_1^2 + x_2^2 + 9x_3^2 - 4x_1x_2 + 12x_1x_3$ and **(b)** $x_1x_2 + x_2x_3 + x_3x_1$ to diagonal forms.

For **(a)**, $4x_1^2 + x_2^2 + 9x_3^2 - 4x_1x_2 + 12x_1x_3$

$= 4\{x_1^2 + x_1(3x_3 - x_2)\} + x_2^2 + 9x_3^2$

$= 4\left\{x_1^2 + 2.x_1.\frac{3x_3 - x_2}{2} + \left(\frac{3x_3 - x_2}{2}\right)^2\right\} + x_2^2 + 9x_3^2 - 4\left(\frac{3x_3 - x_2}{2}\right)^2$

$= 4\left(x_1 + \frac{3x_3 - x_2}{2}\right)^2 + x_2^2 + 9x_3^2 - 9x_3^2 + x_2^2 + 6x_2x_3.$

$= (2x_1 + 3x_3 - x_2)^2 + 6x_2x_3.$

We change the variables as: $x_1 = y_1$, $x_2 = y_2$, and $x_3 = y_2 + y_3$. Then the above expression $(2x_1 + 3x_3 - x_2)^2 + 6x_2x_3$

$= (2y_1 + 3y_2 + 3y_3 - y_2)^2 + 6y_2 (y_2 + y_3)$

$= (2y_1 + 2y_2 + 3y_3)^2 + 6 \left\{y_2^{\,2} + 2y_2 \frac{y_3}{2} + \left(\frac{y_3}{2}\right)^2\right\} - 6 \left(\frac{y_3}{2}\right)^2$

$= (2y_1 + 2y_2 + 3y_3)^2 + 6 \left(y_2 + \frac{y_3}{2}\right)^2 - \frac{6}{4} y_3^{\,2}$

$= (2y_1 + 2y_2 + 3y_3)^2 + \frac{6}{4} \{(2y_2 + y_3)^2 - y_3^{\,2}\}$

Finally changing the variables as $2y_1 + 2y_2 + 3y_3 = z_1$, $2y_2 + y_3 = z_2$ and $y_3 = z_3$, we get the above quadratic form is $z_1^{\,2} + \frac{3}{2}(z_2^{\,2} - z_3^{\,2})$ which is in diagonal form. Here the transformation $Px = z$ is non-singular, because here P is the non-singular the matrix $\begin{pmatrix} 2 & -1 & 3 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{pmatrix}$.

For the **(b)** part the quadratic form is $x_1x_2 + x_2x_3 + x_3x_1$. Here no square term is there and since the 1st non-zero term is x_1x_2, we change the variables to $x_1 = y_1$, $x_2 = y_1 + y_2$ and $x_3 = y_3$. So this form is $y_1 (y_1 + y_2) + (y_1 + y_2)\, y_3 + y_1y_3$

$= y_1^{\,2} + y_1y_2 + y_1y_3 + y_2y_3 + y_1y_3$

$= y_1^{\,2} + y_1(y_2 + 2y_3) + y_2y_3$

$= \left\{y_1^{\,2} + 2.y_1 \left(\frac{y_2 + 2y_3}{2}\right) + \left(\frac{y_2 + 2y_3}{2}\right)^2\right\} - \left(\frac{y_2 + 2y_3}{2}\right)^2 + y_2y_3.$

$= \left(y_1 + \frac{y_2 + 2y_3}{2}\right)^2 - \frac{1}{4}\{y_2^{\,2} + 4y_3^{\,2} + 4y_2y_3\} + y_2y_3.$

$= \frac{1}{4}(2y_1 + y_2 + 2y_3)^2 - \frac{1}{4} y_2^{\,2}y_3^{\,2}.$

Finally replacing $2y_1 + 2y_2 + 3y_3 = z_1$, and $y_2 = z_2$ and $y_3 = z_3$ the above form will reduce to $\frac{1}{4}(z_1^2 - z_2^2) - z_3^2$. Here also the transformation Px = z is non-singular as the matrix P is $\begin{pmatrix} 1 & 1 & 2 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, which is non-singular.

10.3 Classification of Quadratic Forms

Quadratic forms are classified into several categories according to their range. These are given below.

Definition 10.3.1: A quadratic form $Q = x^T A x$ is said to be

- **(i)** Negative definite if $Q < 0$ for $x \neq 0$.
- **(ii)** Negative semi-definite if $Q \leq 0$ for all x and $Q = 0$ for some $x \neq 0$.
- **(iii)** Positive definite if $Q > 0$ for $x \neq 0$.
- **(iv)** Positive semi-definite if $Q \geq 0$ for all x and $Q = 0$ for some $x \neq 0$.
- **(v)** Indefinite if $Q > 0$ for some x and $Q < 0$ for other x.

Since there is one to one correspondence between real symmetric matrices and quadratic forms similar kind of classification is also there for the symmetric matrices. A real symmetric matrix A belongs to a class if the corresponding quadratic form $x^T A x$ belong to the same class.

Example 10.3.1: The form $Q_1 = -x_1^2 - 2x_2^2$ is a negative definite form where as: $Q_2 = -x_1^2 + 2x_1 x + x_2^2$ is a negative semi-definite because $Q_2 = -(x_1 - x_2)^2$ which is always negative and also takes value zero for $x_1 = x_2 \neq 0$. The form $Q_3 = 2x_1^2 +$

$3x_2^2$ is positive definite where as: $Q_4 = x_1^2 - 2x_1x_2 + x_2^2$ is positive semi-definite. Finally $Q_5 = x_1^2 - x_2^2$ is an indefinite form.

10.4 Rank and Signature of a Quadratic Form

To define rank and signature of a quadratic form we use its diagonal representation as given below.

For a real symmetric matrix A let P(A) and N(A) be the numbers of positive and negative diagonal entries in any diagonal form to which $x^TA\,x$ is reduce through a non-singular transformation. The number P(A) – N(A) is called the *signature* of the quadratic form $x^T A\, x$. However rank of the matrix A is called the *rank* of the form $x^T A\, x$.

The quadratic form in example 10.2.2(a) has signature equal to 1 where as that in example 10.2.2(b) has signature – 1.

The classification of quadratic forms can also be done according to their rank and signatures as given in the theorem below.

Theorem 10.4.1: Let $Q = x^T A\, x$ be an n variable quadratic form with rank r and signature s then Q is

- **(i)** Positive definite if and only if $s = n$.
- **(ii)** Positive semi-definite if and only if $r = s$.
- **(iii)** Negative definite if and only if $s = -n$.
- **(iv)** Negative semi-definite if and only if $r = -s$.

(v) Indefinite if and only if $| s | < r$.

The following is an important result on non-singular transformation of quadratic forms.

Theorem 10.4.2: Two quadratic forms on the same number of variables can be obtained from each other through a non-singular transformation if and only if they have the same rank and signature.

10.5 Hermitian Forms

The complex analogue of real quadratic form is known as Hermitian form. Here all vectors as well as matrices are taken as complex.

For a vector x in $\square^n$ and a hermitian matrix A, the expression $\bar{x}^T A x$ is called a *Hermitian form* where $\bar{x}$ is complex conjugate of x. Notice that if x and A are real then Hermitian form will be a quadratic form only.

Although the vector x and the matrix A are complex, the Hermitian form always takes real value that can be seen in the theorem below.

Theorem 10.5.1: A Hermitian form takes real values only.

Proof: Let $H = x^T A x$ be a Hermitian form. Complex conjugate of H is

$$\overline{H} = \overline{\left(\bar{x}^T A x\right)} = \overline{\left(\bar{x}^T\right)}\,\overline{A}\,\bar{x} = x^T\,\overline{A}\,\bar{x}.$$

Since H is a scalar, $H = H^T = \left(\bar{x}^T A\, x\right)^T = x^T A^T \bar{x}$. Since A is Hermitian $\bar{A} = \bar{A}$, so $\bar{H} = x^T\, \bar{A}\, \bar{x} = x^T A^T\, \bar{x} = H^T = H$. Therefore A is real.

Example 10.5.1: Consider a Hermitian matrix A = $\begin{pmatrix} 2 & 3+i \\ 3-i & 1 \end{pmatrix}$. The Hermitian form associated with this is

$$H = \bar{x}^T A x = \left(\overline{x_1}, \overline{x_2}\right)\begin{pmatrix} 2 & 3+i \\ 3-i & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

$= 2x_1\, \overline{x_1} + (3 + i)\, x_2\, \overline{x_1} + (3 - i)\, \overline{x_2}\, x_1 + x_2\, \overline{x_2}.$

$= 2|\, x_1|^2 + 2\,\mathrm{Re}\,\{(3 + i)\, x_2 \overline{x_1}\} + |\, x_2|^2.$

which is a real number.

10.6 Conclusions

Vast literature is there on quadratic forms, to know them on should do further reading. Quadratic forms occur naturally in the study of conics and quadrics in geometry.

Keywords: Quadratic forms, positive definite matrix, negative definite matrix, rank and signature, Hermitian forms.

Suggested Readings:

Linear Algebra, Kenneth Hoffman and Ray Kunze, PHI Learning pvt. Ltd., New Delhi, 2009.

Linear Algebra, A. R. Rao and P. Bhimasankaram, Hindustan Book Agency, New Delhi, 2000.

Linear Algebra and Its Applications, Fourth Edition, Gilbert Strang, Thomson Books/Cole, 2006.

Matrix Methods: An Introduction, Second Edition, Richard Bronson, Academic press, 1991.