Lesson 47

Laplace Equation in 2-Dimensions

47.1 Introduction

Heat conduction in a two dimensional region is given by $\frac{\partial z}{\partial t} = \alpha\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right)$ where $z(t,x,y)$ denoting the temperature in the region. This is clearly a parabolic equation. When we consider steady state conditions, $z = z(x,y)$ i.e, z is independent of time and the equation reduces to $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$ which will be elliptic in nature. Unlike the hyperbolic and parabolic equations where initial conditions are also specified, in case of elliptic equation only boundary conditions are specified, thus making these problems as pure boundary value problems. Let Ω be the interior of a simple closed differentiable boundary curve Γ and f be a continuous function defined on the boundary Γ. The problem of finding the solution of the above Laplace equation in Ω such that it coincides with the function f on the boundary Γ is called the **Dirichlet Problem**.

Finding a function $z(x,y)$ that satisfies the Laplace equation in Ω and satisfies $\frac{\partial z}{\partial \eta} = f(s)$ on Γ where $\frac{\partial}{\partial \eta}$ representing the normal derivative along the outward normal direction to the surface $z(x,y)$ that obeys $\int_\Gamma f(s)ds = 0$ is known as the **Neumann Problem**. The third boundary value problem , known as the **Robin Problem** is one in which the solution of the Laplace equation is obtained in Ω that satisfies the condition $\frac{\partial z}{\partial \eta} + g(s)z(s) = 0$ on Γ where $g(s) \geq 0$ and $g(s) \neq 0$. We now describe the method of Separation of variables technique for the Laplace equation.

We have the Laplace equation given by $\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=0$ (47.1)

Let $z(x,y)=X(x)\ Y(y)$ (47.2)

Finding $\frac{\partial^2 z}{\partial x^2}$ and $\frac{\partial^2 z}{\partial y^2}$ and substituting these in (1) and separating them into two ordinary differential equations, we get

$X''-\lambda X=0$ and $Y''+\lambda Y=0$. where λ is the arbitrary separation parameter. Solving these equations, we get three possible solutions for $\lambda=p^2, \lambda=-p^2$ and $\lambda=0$. These forms are:

a) $z(x,y)=\left(c_1e^{px}+c_2e^{-px}\right)\left(c_3\cos py+c_4\sin py\right);\ \lambda>0$

b) $z(x,y)=\left(c_5\cos px+c_6\sin px\right)\left(c_7e^{py}+c_2e^{-py}\right);\ \lambda<0$

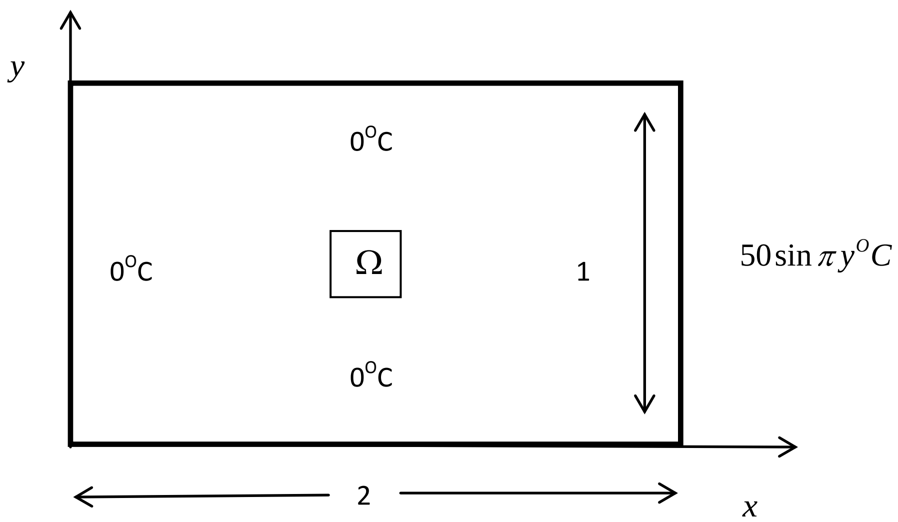
and c) $z(x,y)=\left(c_9x+c_{10}\right)\left(c_{11}y+c_{12}\right);\lambda=0$.

Of these, we take that solution which is consistent with the given boundary conditions.

47.2 Dirichlet Problem in a Rectangular Region:

Example 1:

Solve the Laplace equation $\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=0$ in the region with the boundary conditions as shown in the figure



Solution:

$$\frac{X''}{X} = -\frac{Y''}{Y} = p^2$$

Note: Here we considered $\lambda = p^2$ to allow sinusoidal variation with y, to be consistent with the boundary conditions.

Then the general solution is

$$z(x,y) = \left(Ae^{px} + Be^{-px}\right)\left(C\cos\beta y + D\sin\beta y\right).$$

The boundary conditions are expressed as

$$z(x=0,y) = 0^0C;\, z(x=2,y) = 50\sin\pi y^0C$$

$$z(x,y=0) = 0^0C;\, z(x,y=1) = 0^0C.$$

Now $z(0,y) = 0\forall y \Rightarrow A + B = 0 \Rightarrow A = -B$.

Hence $z(x,0) = 0\forall x \Rightarrow D = 0$.

$$z(x,1) = 0\forall x \Rightarrow \sin\beta = 0 \Rightarrow \beta = \pi$$

$$\therefore z(x,y) = AC\left(e^{\pi x} - e^{-\pi x}\right)\sin\pi y$$

The non-homogeneous boundary condition at $x = 2$

$$\Rightarrow AC\left(e^{2\pi} - e^{-2\pi}\right)\sin\pi y = 50\sin\pi y$$

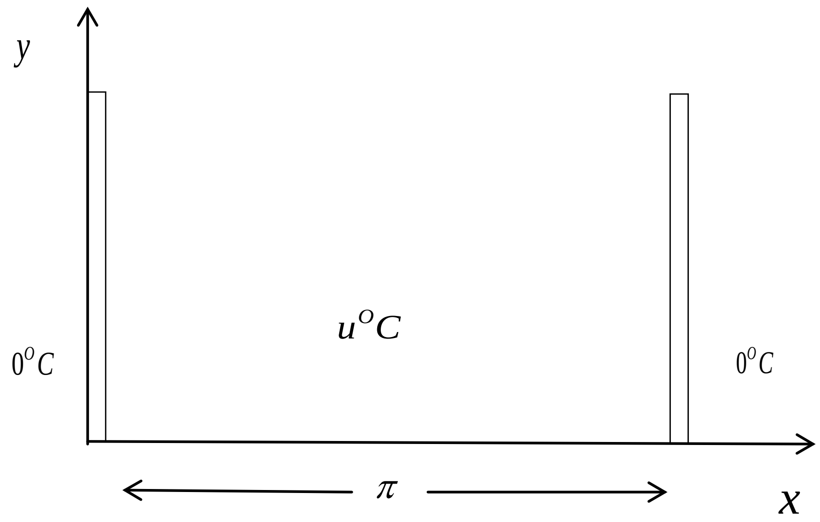
$$\Rightarrow AC = \frac{50}{\left(e^{2\pi} - e^{-2\pi}\right)} = 0.0934;(x,y)$$

Thus the temperature at any point (x,y) is written as $z(x,y) = 0.0934\left(e^{\pi x} - e^{-\pi x}\right)\sin\pi y$.

47.3 Temperature Distribution is Studied in an Infinitely Long Plate.

Example 2:

An infinitely long plane uniform plate is bounded by two parallel edges and at an end at right angles to them as shown in the adjacent figure. Find the temperature distribution at any point of the plate in the steady state.



Solution:

The steady state temperature distribution in this infinitely long plate is obtained by solving $\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=0$.

The boundary conditions are $z(x=0,y)=0^0C$, $z(x=\pi,y)=0^0C\forall y>0$,

$z(x,y=0)=u_0{}^0C$ for $0<x<\pi$, $z(x,y\to\infty)\to 0$ for $0<x<\pi$.

Among the three possibilities for solution i.e., solution forms (a), (b), (c), we chose a solution that is consistent with the given boundary conditions. He solution given in equation (a) cannot satisfy the boundary condition $z(0,y)=0\forall y$. The solution given in equation (c) cannot satisfy the condition in $z(x,y\to\infty)\to 0$ in $0<x<\pi$.

Thus we have the solution as $z(x,y) = \left(A\cos px + B\sin px\right)\left(Ce^{py} + De^{-py}\right)$.

Now $z(0,y) = A\left(Ce^{py} + De^{-py}\right) = 0 \Rightarrow A = 0$

$\therefore z(x,y) = B\sin px\left(Ce^{py} + De^{-py}\right)$.

$z(\pi,y) = 0 \forall y \quad \Rightarrow B\sin p\pi\left(Ce^{py} + De^{-py}\right) = 0 \Rightarrow p = n$, an integer $\left(\because B \neq 0\right)$.

$\therefore z(x,y) = B\sin nx\left(Ce^{py} + De^{-py}\right)$.

As $z(x, y \to \infty) \to 0 \Rightarrow c = 0 \quad \therefore z(x,y) = BD\sin nxe^{-ny}$.

Taking $BD = b_n$ and write the general form of the solution as $z(x,y) = \sum_{n=1}^{\infty} b_n \sin nxe^{-ny}$.

Using the non-homogeneous boundary condition $u(x,0) = u_0 = \sum_{n=1}^{\infty} b_n \sin nx$

The unknown coefficients are found using the half range Fourier sine series expansion in $(0,\pi)$ as

$$b_n = \frac{2}{\pi}\int_0^{\pi} u_0 \sin nxdx = \frac{2}{\pi}u_0\left[1-(-1)^n\right] = \begin{cases} \dfrac{4u_0}{n\pi}, n = 2m-1 \\ 0, n = 2m \end{cases}$$, m is a positive integer.

Thus $z(x,y) = \dfrac{4u_0}{\pi}\left[e^{-y}\sin x + \dfrac{1}{3}e^{-3y}\sin 3x + \ldots\right]$.

Example 3

Solve $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = 0$ subject to $z(0,y) = z(a,y) = z(x,b) = 0$ and $z(x,0) = z(a-x), 0 < x < a$.

Solution

Physically realistic solution here is

$$z(x,y)=\left(c_1\cos px+c_2\sin px\right)\left(c_3e^{py}+c_4e^{-py}\right).$$

$$z(0,y)=0\Rightarrow c_1=0,$$

$$z(a,y)=0\Rightarrow \sin pa=0\Rightarrow p=\frac{n\pi}{a},\ n \text{ is an integer}$$

$$\therefore z(x,y)=c_2\sin\frac{n\pi x}{a}\left(c_3e^{\frac{n\pi y}{a}}+c_4e^{\frac{-n\pi y}{a}}\right)$$

Take $c_2c_3=A$, $c_2c_4=B$.

$$z(x,b)=0\Rightarrow Ae^{\frac{n\pi y}{a}}+Be^{\frac{-n\pi y}{a}}=0 \qquad \Rightarrow A=\frac{-B\exp\left(\frac{-n\pi y}{a}\right)}{\exp\left(\frac{n\pi y}{a}\right)}.$$

$$\therefore z(x,y)=\sin\frac{n\pi x}{a}\left[\frac{-Be^{\frac{-n\pi b}{a}}}{e^{\frac{n\pi b}{a}}}e^{\frac{n\pi y}{a}}+Be^{\frac{-n\pi y}{a}}\right]$$

$$=\frac{-B}{e^{\frac{n\pi b}{a}}}\sin\frac{n\pi x}{a}\left[e^{\frac{n\pi(y-b)}{a}}-e^{\frac{-n\pi(y-b)}{a}}\right].$$

So the general solution is now written as

$$z(x,y)=\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{a}\sinh\frac{n\pi\left(y-b\right)}{a},\quad \text{where } b_n=\frac{-2B}{e^{\frac{n\pi b}{a}}}$$

Now using the non-homogeneous condition

$$z(x,0)=x(a-x)=\sum_{n=1}^{\infty}b_n\sinh\frac{n\pi b}{a}\sin\frac{n\pi x}{a}\quad=\sum_{n=1}^{\infty}B_n\sin\frac{n\pi x}{a}\ \text{(say)}$$

the coefficient B_n are found as $B_n = \frac{2}{a}\int_0^x x(a-x)\sin\frac{n\pi x}{a}dx$

$$= \frac{4a^2}{n^3\pi^3}(1-\cos n\pi) = \begin{cases} \frac{8a^2}{n^3\pi^3}, n = 2m-1 \\ 0, n = 2m \end{cases}$$, m is a positive integer.

$$\therefore z(x,y) = \frac{8a^2}{\pi^3}\sum_{n=1,3,5,...}\frac{\sin\frac{4n\pi(b-y)}{a}}{n^3\sinh\frac{n\pi(b-y)}{a}}\sin\frac{n\pi x}{a}$$

or $z(x,y) = \frac{8a^2}{\pi^3}\sum_{n=0}^{\infty}\frac{\sinh\frac{(2n+1)\pi(b-y)}{a}}{(2n+1)^3\sinh\frac{(2n+1)\pi(b-y)}{a}}\sin\frac{\left(2n+1\right)\pi x}{a}$.

Keywords: Dirichlet Problem, Neumann Problem, Robin Problem

Exercises 1

1. Solve $\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=0$ in $0<x<\pi, 0<y<\pi;$ with the conditions $z(0,y)=z(\pi,y)=z(x,\pi)=0$; $z(x,0)=\sin^2 x$

2. A rectangular plate has sides a and b. taking the side of length a as OX and that of length b as OY and other sides to be $x=a$ and $y=b$, the sides $x=0$, $x=a$, $y=b$ are insulated and the edge $y=0$ is kept at temperature $u_0\cos\frac{\pi x}{a}$. Find the temperature $z(x,y)$ in the steady state.

3. Solve $\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=0$ subject to

(i) $z(0, y) = 0; z(x, 0 = 0);\ z(1, y) = 0; z(x,1) = 100\sin \pi x$.

(ii) $z(0, y) = 0; z(x, 0 = 0);\ z(1, y) = 100\sin \pi y; z(x,1) = 0$.

(iii) $z(0, y) = 0; \frac{\partial z}{\partial y}(x, 0) = 0;\ \frac{\partial z}{\partial x}(1, y) = 0; z(x,1) = 100$.

(iv) $z(0, y) = 100; z(x, 0) = 100;\ z(1, y) = 200; z(x,1) = 100$.

References

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Suggested Reading

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