Lesson 16

Stirling's and Bessel's Formula

16.1 Stirling's Formula

This is obtained by taking the mean of the Gauss Forward and Backward interpolation formulae.

This is written as:

$$y_p = y_0 + p\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + \frac{p^2}{2!}\Delta^2 y_{-1} + \frac{p\left(p^2-1\right)}{3!}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \frac{p^2\left(p^2-1\right)}{4!}\Delta^4 y_{-2} + \ldots \tag{16.1}$$

Writing this using central differences, we obtain

$$y_p = y_0 + \frac{p}{2}\left(\delta y_{\frac{1}{2}} + \delta y_{-\frac{1}{2}}\right) + \frac{p^2}{2!}\delta y_0 + \frac{p\left(p^2+1^2\right)}{3!\cdot 2}\left(\delta^3 y_{\frac{1}{2}} + \delta^3 y_{-\frac{1}{2}}\right) + \frac{p^2\left(p^2-1^2\right)}{4!}\delta^4 y_0 + \ldots \tag{16.2}$$

This is called the Stirling's formula.

Example 1: Find the value of e^x when $x = 0.644$ from the below given table:

x	0.61	0.62	0.63	0.64	0.65
$y = e^x$	1.840431	1.858928	1.87761	1.896481	1.91554

Solution:

$$x = 0.644, x_0 = 0.64, p = \frac{0.644 - 0.64}{0.01} = 0.4, y_0 = 1.896481.$$

By forming the difference table (left as an exercise!) we note that $\Delta y_{-1} = 0.018871, \Delta y_0 = 0.01906, \Delta^2 y_{-1} = 0.000189$ and all higher order differences are approximately zero. Substituting these in the Stirling's formula given in (1), we get

$y(0.64) = 1.896481 + 0.0075862 + 0.00001512 = 1.904082$.

16.2 Bessel's Formula

We know $\Delta^2 y_0 - \Delta^2 y_{-1} = \Delta^3 y_{-1}$

$\Rightarrow \Delta^2 y_{-1} = \Delta^2 y_0 - \Delta^3 y_{-1}$.

Similarly $\Delta^4 y_{-1} - \Delta^4 y_{-2} = \Delta^5 y_{-2}$

$\Rightarrow \Delta^4 y_{-2} = \Delta^4 y_{-1} - \Delta^5 y_{-2}$

Using these in the Gauss forward interpolation formula, we obtain

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\left(\frac{1}{2}\Delta^2 y_{-1} + \frac{1}{2}\Delta^2 y_{-1}\right) + \frac{p(p^2-1)}{3!}\Delta^3 y_{-1}$$

$$+\frac{p(p^2-1)(p-2)}{4!}\left(\frac{1}{2}\Delta^4 y_{-2} + \frac{1}{2}\Delta^4 y_{-2}\right) + \ldots$$

$$= y_0 + p\Delta y_0 + \frac{1}{2}\frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{1}{2}\frac{p(p-1)}{2!}\left(\Delta^2 y_0 - \Delta^3 y_{-1}\right) + \frac{p(p^2-1)}{3!}\Delta^3 y_{-1}$$

$$+\frac{1}{2}\frac{p\left(p^2-1\right)\left(p-2\right)}{4!}\Delta^4 y_{-2}+\frac{1}{2}\frac{p\left(p^2-1\right)\left(p-2\right)}{4!}\left(\Delta^4 y_{-1}-\Delta^5 y_{-2}\right)+\ldots$$

$$\text{or } y_p = y_0 + p\Delta y_0 + \frac{p\left(p-1\right)}{2!}\left(\frac{\Delta^2 y_{-1}+\Delta^2 y_0}{2}\right)+\frac{\left(p-\frac{1}{2}\right)p\left(p-1\right)}{3!}\Delta^3 y_{-1}$$

$$+\frac{\left(p+1\right)p\left(p-1\right)\left(p-2\right)}{4!}\left(\frac{\Delta^4 y_{-2}+\Delta^4 y_{-1}}{2}\right)+\ldots \tag{16.3}$$

This is known as the Bessel's formula.

Example 2: Using Bessel's formula, obtain $y(25)$, given $(20) = 2854, y(24) = 3162, y(28) = 3544, y(32) = 3992$.

Solution:

Taking $x_0 = 24, h = 4, y_0 = 3162$.

We have $p = \frac{1}{4}\left(x-24\right)$.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
20	2854			
		308		
24	3162		74	
		382		-8
28	3544		66	
		448		

32 3992

Taking $x = 25, p = \frac{1}{4}$.

Bessel's formula is:

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\left(\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2}\right) + \frac{\left(p - \frac{1}{2}\right)p(p-1)}{3!}\Delta^3 y_{-1} + \ldots$$

$$\therefore y(25) = 3162 + (0.25)(382) + \frac{(0.25)(-0.75)}{2}\left(\frac{74+66}{2}\right) + \frac{(-0.25)(0.25)(-0.75)}{6}(-8)$$

$= 3250.87$.

Note:

1. If the value of p lies between $-\frac{1}{4}$ and $\frac{1}{4}$, prefer Stirling's formula, it gives a better approximation.
2. If p lies between $\frac{1}{4}$ and $\frac{3}{4}$, Bessel's or Everett's formula gives better approximation.

<u>Exercises</u>:

1. Using Stirling's formula, find $y(35)$ from the data $y(20) = 512, y(30) = 439, y(40) = 346, y(50) = 243$.

2. Find $f(34)$ using Bessel's formula from

x:	20	25	30	35	40
$f(x)$:	11.47	12.78	13.76	14.49	15.05

3. Tabulate $f(x) = e^{-x}$ in $[1.72, 1.78]$ with $h = 0.01$. Find $f(1.7475)$ using (i). Bessel's and (ii) Everett's formula.

Keywords: Bessel's Formula, Stirling's Formula

References

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Suggested Reading

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