Lesson 31

Methods for Solving Higher Order Initial Value Problems

31.1 Introduction

From the Theory of ordinary differential equations, it is evident that an n^{th}order ordinary differential equation, be a linear or a non-linear one, can be reduced to a system of n-first order equations. To see this, consider the second order o. d. e.

$y'' - y' + 4t^2 y = 0$

subject to the initial conditions $y(1) = 1$, $y'(1) = 2$.

Let $u = y(t)$ and $v = y'(t)$,

then $v' = y''$

and we have $u' = v$ and $v' = v - 4t^2 u$.

Thus the 2-first order equations are

$u' = v$; $u(1) = 1$

$v' = v - 4t^2 u$; $v(1) = 2$

This is known as the initial value problem in the first order system corresponding to the given 2nd order initial value problem.

In general an n^{th}order differential equation

$$y^{(n)} = F(t, y, y', y'', ..., y^{n-1}) \tag{31.1}$$

is written in the first order system as follows set $u_1 = y(t)$.

$$\left.\begin{aligned} u_1' &= u_2 \\ u_2' &= u_3 \\ &\dots \\ u_{n-1}' &= u_n \\ u_n' &= F(t, u_1, u_2, \dots, u_{n-1}) \end{aligned}\right\} \tag{31.2}$$

With the transformed initial conditions

$$u_1(t_0) = \eta_0, u_2(t_0) = \eta_1 \dots, u_n(t_0) = \eta_{n-1} \tag{31.3}$$

This system is written in the vector form as

$$\left.\begin{aligned} \underline{u}' &= \underline{f}(t, \underline{u}) \\ \underline{u}(t_0) &= \underline{\eta} \end{aligned}\right\}$$

(31.4)

where $\underline{u} = [u_1, u_2, \dots, u_n]^T$,

$\underline{f} = [u_2, u_3, \dots, u_n, F]^T$,

$\underline{\eta} = [\eta_0, \eta_1, \dots, \eta_{n-1}]^T$.

Thus the methods of solution of the first order I.V.P

$$\frac{dy}{dt} = f(t, y), y(t_0) = y_0$$

can be used to solve the above system of first order I.V.Ps.

31.2 Taylor Series Method

In what follows is shown the utility of Taylor series method to the system of I.V.Ps. through two examples.

1. The vector form of the Taylor Series method is written as

$$\underline{y}_{j+1} = \underline{y}_j + h\underline{y}'_j + \frac{h^2}{2!}\underline{y}''_j + \ldots + \frac{h^p}{p!}\underline{y}_j^{(p)}$$

Here $j = 0,1,2,..,N-1$ denote the nodal point

and $\underline{y}_j^{(k)} = \begin{bmatrix} y_{1,j}^{(k)} \\ y_{2,j}^{(k)} \\ \ldots \\ y_{n,j}^{(k)} \end{bmatrix} = \begin{bmatrix} \frac{d}{dt^{k-1}} f_1(t_j, y_{1,j}, y_{2,j}, \ldots, y_{n,j}) \\ \ldots \\ \frac{d^{k-1}}{dt^{k-1}} f_n(t_j, y_{1,j}, y_{2,j}, \ldots, y_{n,j}) \end{bmatrix}$.

31.3 Euler Method

The vector form of Euler method can be written as:

$\underline{y}_{j+1} = \underline{y}_j + h\underline{y}'_j$, $j = 0,1,2,\ldots,N-1$.

Example: 1. Reduce the 3rd order I.V.P. into a system of first order I.V.P.:

$y''' + 2y'' + y' - y = \cos t, t \in [0,1]$

subject to $y(0) = 0, y'(0) = 1, y''(0) = 2$.

Solution:

Set $y = u_1, u'_1 = u_2, u'_2 = u_3$.

Now the system of 3 first order equations is

$u'_1 = u_2$

$u'_2 = u_3$

$u'_3 = \cos t - 2u_3 - u_2 + u_1$

The initial conditions are: $u_1(0) = 0, u_3(0) = 1, u_3(0) = 2$.

Example 2. Use the 2nd order Taylor Series method to compute $y(1), y'(1)$ and $y''(1)$ by taking $h = 1.0$ in the above example.

Solution:

The Second order Taylor Series method is

$$\underline{u}(t_0 + h) = \underline{u}(t_0) + h\underline{u}'(t_0) + \frac{h^2}{2}\underline{u}''(t_0)$$

Given $h = 1$

$$\therefore \underline{u}(1) = \underline{u}(0) + \underline{u}'(0) + \frac{1}{2}\underline{u}''(0).$$

The system of I.V.P. is:

$$\underline{u}' = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}' = \begin{bmatrix} u_1 \\ u_3 \\ \cos t - 2u_3 - u_2 + u_1 \end{bmatrix}$$

subject to $\underline{u}(0) = \begin{bmatrix} u_1(0) \\ u_2(0) \\ u_3(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$.

We now require to compute $\underline{u}'(0)$ and $\underline{u}''(0)$:

$$\underline{u}'(0) = \begin{bmatrix} u_2(0) \\ u_3(0) \\ 1 - 2u_3(0) - u_2(0) + u_1(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -4 \end{bmatrix} = \begin{bmatrix} u_1'(0) \\ u_2'(0) \\ u_3'(0) \end{bmatrix}$$

Also $\underline{u}''(0) = \begin{bmatrix} u_2'(0) \\ u_3'(0) \\ -2u_3'(0) - u_2'(0) + u_1'(0) \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \\ 7 \end{bmatrix}$

$$\therefore \underline{u}(1) = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ -4 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 2 \\ -4 \\ 7 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ \frac{3}{2} \end{bmatrix} = \begin{bmatrix} y(1) \\ y'(1) \\ y''(1) \end{bmatrix}.$$

Exercises:

1. Solve the system equations $u' = -3u + 2v, u(0) = 0$ and $v' = 3u - 4v, v(0) = \frac{1}{2}$ using

(i) Forward Euler method and

(ii) 2nd order Taylor Series method by taking $h = 0.2$ on the interval $[0, 0.6]$.

Keywords: Euler Method, Higher order initial value problems, Taylor Series method.

References

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Suggested Reading

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