Lesson 30

Linear Differential Equations of Higher Order

In this lesson we discuss linear differential equation of higher order with constant coefficients. In particular, we shall learn about the techniques of finding solutions of homogenous equations. Different cases will be considered with the help of several examples.

30.1 Linear Differential Equation

In a linear differential equation, the dependent variable and its differential coefficients occur only in the first degree and are not multiplied together. The general form of the equation is

$$\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + a_2(x)\frac{d^{n-2}y}{dx^{n-2}} + \ldots + a_n(x)y = F(x), \tag{30.1}$$

where $a_1, a_2, \ldots, a_n$ and F are either constants or functions of x only. If the right hand side, i.e. $F(x)$, is identically zero, the equation is said to be homogeneous; otherwise it is called nonhomogeneous. Before we discuss some particular cases of the above equation we state two facts about the solution of a linear homogeneous differential equation. The first says that if we know n solutions $y_1, y_2, \ldots, y_n$ of the linear homogeneous equation, then any linear combination $y = c_1y_1 + c_2y_2 + \ldots c_ny_n$ is also a solution for any constants $c_1, c_2, \ldots, c_n$. This can easily be proved by substituting $y = c_1y_1 + c_2y_2 + \ldots c_ny_n$ into the equation and using linearity of the equation. The second important result concerns about the general solutions (solution containing all solutions) to the linear homogeneous equation. This result says that any solution is some linear combination of $y_1, y_2, \ldots, y_n$ for some suitable values of constants $c_1, c_2, \ldots, c_n$. However, this is not true for any combination of solutions but is true if the solutions $y_1, y_2, \ldots, y_n$ are linearly independent.

30.2 Linear Differential Equation with Constant Coefficients

An equation of the form

$$\frac{d^n y}{dx^n} + a_1\frac{d^{n-1}y}{dx^{n-1}} + a_2\frac{d^{n-2}y}{dx^{n-2}} + \ldots + a_ny = F(x), \tag{30.2}$$

where $a_1, a_2, \ldots, a_n$ are constants, is called linear differential equation with constant coefficients. Using the symbols $D^n := \dfrac{d^n}{dx^n}$, the Equation (30.2) becomes

$$(D^n + a_1 D^{n-1} + a_2 D^{n-2} + ... + a_n)y = F(x), \tag{30.3}$$

Further defining $f(D) := D^n + a_1 D^{n-1} + a_2 D^{n-2} + ... + a_n$, we can rewrite the given differential equation in a more compact form as $f(D)y = F(x)$. Here $f(D)$ acts as operator on y to yield $F(x)$. The general solution of (30.2) can be written as the sum of the general solution of the corresponding homogeneous equation, refereed as complimentary function (C.F.), and a particular solution or sometimes called particular integral (P.I) of nonhomogeneous equation. Thus

$$y = C.F. + P.I. \tag{30.4}$$

Note that the C.F. involves n arbitrary constants and P.I. does not involve any arbitrary constant. It is readily evident that y in (30.4) is the general solution of the given nonhomogeneous differential equation because it satisfies the given differential equation as $f(D)\,(C.F. + P.I.) = f(D)\,(C.F.) + f(D)\,(P.I.) = 0 + F(x)$ and it has n arbitrary constants.

30.3 C.F. of a Differential Equation

By definition, C.F. of (30.2) is the general solution of

$$(D^n + a_1 D^{n-1} + a_2 D^{n-2} + ... + a_n)y = 0 \tag{30.5}$$

To solve Equation (30.5), we seek a function which satisfies the above equation. One intelligent guess of such a function is the exponential function e^{mx}, where m is a constant. Differentiations of this exponential function are just constant multiples of the original exponential. If we substitute this function into the Equation (30.5), we obtain

$$(m^n + a_1 m^{n-1} + a_2 m^{n-2} + ... + a_n)e^{mx} = 0 \tag{30.6}$$

Since the exponential function is never zero, we can divide this last equation by e^{mx}. Thus, $y = e^{mx}$ is a solution to Equation (30.5) if and only if m is a solution to the algebraic equation

$$m^n + a_1 m^{n-1} + a_2 m^{n-2} + ... + a_n = 0 \tag{30.7}$$

Equation (30.7) is called the auxiliary equation (A.E.) or characteristic equation (C.E.) of the differential Equation (30.5).

30.4 Case I: A.E. has real and distinct roots

If $m_1, m_2, m_3, ..., m_n$ be real and distinct then the solutions $e^{m_1 x}, e^{m_m x}, \ldots, e^{m_n x}$ are linearly independent and the general solution of the given homogeneous differential equation becomes

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x} + ... + c_n e^{m_n x},$$

where $c_1, c_2, ..., c_n$ are arbitrary constants.

30.4.1 Example

Find the general solution of the differential equation $(D^3 + 6D^2 + 11D + 6)y = 0$.

Solution: The A.E. is $(m^3 + 6m^2 + 11m + 6) = 0$. The roots are $m = -1, -2, -3$. Hence the required solution is $y = c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{-3x}$.

30.5 Case II: A.E. has repeated real roots

Let $m_1 = m_2$ are repeated roots of the A.E. Then, we have $n - 1$ linearly independent solutions. It can be shown that a simple choice $y = xe^{m_1 x}$ is also a solution which is independent to the rest $n - 1$ solutions. Thus, the general solution of the given differential equation is given by

$$y = (c_1 + c_2 x)e^{m_1 x} + c_3 e^{m_3 x} + ... + c_n e^{m_n x}$$

The above idea can be further extended by taking solutions $xe^{m_1 x}, x^2 e^{m_1 x}, \ldots, x^{l-1} e^{m_1 x} \ldots$ if the root m_1 is repeating $l-$times.

30.5.1 Example

Find the general solution to $(D^4 + 2D^3 - 3D^2 - 4D + 4)y = 0$.

Solution: The A.E. of given equation is

$$(m^4 + 2m^3 - 3m^2 - 4m + 4) = 0$$

The roots of the A.E. are $m = 1, 1, -2, -2$. The required solution is $y = (c_1 + c_2 x)e^x + (c_3 + c_4 x)e^{-2x}$.

30.6 Case III: A.E. has complex roots

If $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$, then the solutions $e^{m_1 x}, e^{m_2 x}, \ldots, e^{m_n x}$ are linearly independent and the general solution of the given homogeneous differential equation is given by

$$y = c_1' e^{m_1 x} + c_2' e^{m_2 x} + c_3 e^{m_3 x} + ... + c_n e^{m_n x}.$$

The above solution can be simplified as

$$y = c_1' e^{\alpha x} (\cos \beta x + i \sin \beta x) + c_2' e^{\alpha x} (\cos \beta x - i \sin \beta x) + c_3 e^{m_3 x} + ... + c_n e^{m_n x}.$$

Defining new constants $c_1 = c_1' + c_2'$ and $c_2 = i(c_1 - c_2)$, the general solution becomes

$$y = e^{\alpha x} (c_1 \cos \beta x + c_2 \sin \beta x) + c_3 e^{m_3 x} + ... + c_n e^{m_n x}.$$

Similar to the case II, the solution for repeated complex roots can be found, see example below.

30.6.1 Example

Find the general solution to the differential equation $(D^2 + 1)^2 y = 0.$

Solution: The A.E. and its roots are

$$(m^2 + 1)^2 = 0, \quad \text{and therefore} \quad m = \pm i, \pm i.$$

This is the case of repeated complex root, so case II and case III can be combined to give the desired solution as $y = (c_1 + c_2 x) \cos x + (c_3 + c_4 x) \sin x$.

30.7 Miscellaneous Problems

30.7.1 Problem 1

Find the general solution of the differential equation $(D^3 + 3D^2 + 3D + 1)y = 0.$

Solution: The A.E. and its root are given by $(m+1)^3 = 0$ and $m = -1, -1, -1$. Therefore, the required solution is $y = (c_1 + c_2 x + c_3 x^2)e^{-x}$.

30.7.2 Problem 2

Find the general solution of $(D^3 - 8)y = 0$.

Solution: The A.E. of the given equation is $(m^3 - 8) = 0$. Its root are $m = 2, -1 \pm i\sqrt{3}$. The required solution is $y = c_1 e^{2x} + e^{-x}(c_2 \cos \sqrt{3}x + c_3 \sin \sqrt{3}x)$.

30.7.3 Problem 3

Find the general solution of the differential equation $(D^2 - 2D + 5)^2 y = 0$.

Solution: The auxiliary equation is $(m^2 - 2m + 5)^2 = 0$. Its roots are $m = 1 \pm 2i, 1 \pm 2i$ Hence the required solution is $y = e^x \left[(c_1 + c_2 x) \cos 2x + (c_3 + c_4 x) \sin 2x\right]$.

30.7.4 Problem 4

Find the general solution of $(D^2 + D + 1)^2 (D - 2)y = 0$.

Solution: The A.E. of the given equation is $(m^2 + m + 1)^2 (m - 2) = 0$. Its roots are $m = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}, -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}, 2$. Hence, the desired solution is

$$y = c_1 e^{2x} + e^{-\frac{1}{2}x} \left[(c_2 + c_3 x) \cos \frac{\sqrt{3}}{2}x + (c_4 + c_5 x) \sin \frac{\sqrt{3}}{2}x\right]$$

30.7.5 Problem 5

Find the general solution of the differential equation $(D^2 + 1)^3 (D^2 + D + 1)^2 y = 0$.

Solution: The A.E. of given equation is

$$(D^2 + 1)^3 (D^2 + D + 1)^2 = 0,$$

The roots are $m = \pm i, \pm i, \pm i, -\dfrac{1}{2} \pm i\dfrac{\sqrt{3}}{2}, -\dfrac{1}{2} \pm i\dfrac{\sqrt{3}}{2}$. Therefore, the desired solution is

$$\begin{aligned} y =& (c_1 + c_2 x + c_3 x^2) \cos x + (c_4 + c_5 x + c_6 x^2) \sin x \\ &+ e^{-\frac{1}{2}x} \left[(c_7 + c_8 x) \cos \frac{\sqrt{3}}{2}x + (c_9 + c_{10} x) \sin \frac{\sqrt{3}}{2}x\right] \end{aligned}$$

Suggested Readings

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