Lesson 34

Method of Variation of Parameters

In the last lesson we have discussed method of undetermined coefficients for finding particular integral. In this lesson we lean another rather general method, called method of variation of parameters, of finding particular integral of non-homogeneous differential equation. In contrast to the method of undetermined coefficients, this method is also applicable for solving linear equations with variable coefficients. For the sake of simplicity we restrict ourselves for second order linear differential equations. However the method is also applicable for higher order linear differential equations.

34.1 Method of Variation of Parameters

Consider a second order differential equation of the form

$$y'' + Py' + Qy = R \tag{34.1}$$

where P, Q, R are functions of x or constants. If u and v are two linearly independent solutions of the corresponding homogeneous differential equation

$$y'' + Py' + Qy = 0 \tag{34.2}$$

Then, the complimentary function is

$$y = au + bv \tag{34.3}$$

where a, b are two arbitrary constants and u, v are functions of x. Since u and v are solutions of (34.2), we have

$$u'' + Pu' + Qu = 0, \quad v'' + Pv' + Qv = 0. \tag{34.4}$$

The method of variation of parameters relies on finding a particular integral of nonhomogeneous equation by replacing constants a and b with functions of x. The aim is to find functions $A(x)$ and $B(x)$ such that

$$y_p = Au + Bv \tag{34.5}$$

is a particular integral of (34.1). To determine $A(x)$ and $B(x)$ we need to have two equations. These are obtained as follows. First we compute

$$y_p' = Au' + Bv' + A'u + B'v \tag{34.6}$$

In order to avoid second order derivatives of A and B and to simplify the above expression we take

$$A'u + B'v = 0, \tag{34.7}$$

Now, the Equation (34.6) reduces to

$$y_p' = Au' + Bv' \tag{34.8}$$

Differentiating (34.8), we obtain

$$y_p'' = A'u' + Au'' + B'v' + Bv'' \tag{34.9}$$

Using the values of y, y' and y'' given by (34.5), (34.8) and (34.9) into the Equation (34.1), we get

$$A'u' + Au'' + B'v' + Bv'' + P(Au' + Bv') + Q(Au + Bv) = R$$

Further simplifications lead to

$$A(u'' + Pu' + Qu) + B(v'' + Pv' + Qv) + A'u' + B'v' = R$$

Using Equation (34.4) we get

$$A'u' + B'v' = R \tag{34.10}$$

Solving (34.7) and (34.10) for A' and B', we get

$$A' = \frac{-vR}{W}, \quad \text{and} \quad B' = \frac{uR}{W} \tag{34.11}$$

where W is Wronskian of u and v, and given by $W = uv' - u'v \neq 0$. Note that the Wronskian is nonzero because u and v are two linearly independent solutions. Integrating (34.11), we get

$$A = f(x), \quad B = g(x), \tag{34.12}$$

where

$$f(x) = -\int \frac{vR}{W}dx, \quad g(x) = \int \frac{uR}{W}dx$$

Using (34.12) into (34.5), we have

$$y_p = uf(x) + vg(x).$$

Hence, the general solution of the given differential equation is

$$y_p = au + bv + uf(x) + vg(x).$$

34.2 Example Problems

34.2.1 Problem 1

Solve the differential equation $y'' + n^2y = \sec nx$.

Solution: Comparing the given equation with the standard equation $y'' + Py' + Qy = R$, we get $P = 0$, $Q = n^2$ and $R = \sec nx$. The characteristic equation of the corresponding homogeneous equation is

$$(m^2 + n^2)y = 0, \text{ so that } m = \pm in$$

The complimentary function is

$$\text{C.F.} = (c_1 \cos nx + c_2 \sin nx)$$

In order to find a particular integral, we have $u = \cos nx$ and $v = \sin nx$ and $R = \sec nx$. The Wronskian is given as

$$W = \begin{vmatrix} \cos nx & \sin nx \\ -n\sin nx & n\cos nx \end{vmatrix} = n \neq 0.$$

Then, the particular integral of the given equation is

$$\text{P.I.} = uf(x) + vg(x)$$

where

$$f(x) = -\int \frac{vR}{W}dx = -\int \frac{\sin nx \sec nx}{n}dx = \frac{1}{n^2}\ln(\cos nx)$$

and

$$g(x) = \int \frac{uR}{W} dx = -\int \frac{\cos nx \sec nx}{n} dx = \frac{x}{n}$$

Hence, the required solution is

$$y = (c_1 \cos nx + c_2 \sin nx) + \cos nx \frac{1}{n^2} \ln(\cos nx) + \frac{x}{n} \sin nx$$

34.2.2 Problem 2

Find the general solution of the differential equation $y'' + n^2 y = \tan nx$.

Solution: We compare the given equation with $y'' + Py' + Qy = R$ to have $P = 0$, $Q = n^2$ and $R = \tan nx$. Similar to the previous example, we have the complimentary function as

$$\text{C.F.} = (c_1 \cos nx + c_2 \sin nx)$$

To find particular integral we have $u = \cos nx$, $v = \sin nx$, $R = \sec nx$. The Wronskian is given by

$$W = \begin{vmatrix} \cos nx & \sin nx \\ -n \sin nx & n \cos nx \end{vmatrix} = n \neq 0.$$

The particular integral is

$$\text{P.I.} = uf(x) + vg(x)$$

where

$$f(x) = -\int \frac{vR}{W} dx = -\int \frac{\sin nx \tan nx}{n} dx = \frac{1}{n^2}[\sin nx - \ln(\sec nx + \tan nx)]$$

and

$$g(x) = \int \frac{uR}{W} dx = -\int \frac{\cos nx \tan nx}{n} dx = -\frac{1}{n^2} \cos nx$$

The desired general solution is

$$y = (c_1 \cos nx + c_2 \sin nx) + \frac{\cos nx}{n^2}[\sin nx - \ln(\sec nx + \tan nx)] - \frac{1}{n^2} \sin nx \cos nx$$

34.2.3 Problem 3

Solve the differential equation $\dfrac{d^2y}{dx^2} + n^2 y = \cot nx$.

Solution: Similar to the previous example, the complimentary function is given by

$$\text{C.F.} = (c_1 \cos nx + c_2 \sin nx)$$

In this case, we have $u = \cos nx$, $v = \sin nx$ and $R = \cot nx$. The Wronskian is given by

$$W = \begin{vmatrix} \cos nx & \sin nx \\ -n \sin nx & n \cos nx \end{vmatrix} = n \neq 0$$

Then, the particular integral is

$$\text{P.I.} = uf(x) + vg(x)$$

where

$$f(x) = -\int \frac{vR}{W} dx = -\int \frac{\sin nx \cot nx}{n} dx = -\frac{1}{n^2} \sin nx$$

and

$$g(x) = \int \frac{uR}{W} dx = -\int \frac{\cos nx \cot nx}{n} dx = \frac{1}{n^2} \left[\cos nx + \ln\left(\tan \frac{nx}{2}\right)\right]$$

The required solution is

$$y = (c_1 \cos nx + c_2 \sin nx) - \frac{1}{n^2} \cos nx \sin nx + \frac{1}{n^2} \left[\cos nx + \ln\left(\tan \frac{nx}{2}\right)\right] \sin nx$$

34.2.4 Problem 4

Using the method of variation of parameters, find the general solution of the differential equation

$$\frac{d^2y}{dx^2} + y = \sec^2 x$$

Solution: The complimentary function is given by

$$\text{C.F.} = (c_1 \cos x + c_2 \sin x)$$

Also we have $u = \cos x$ and $v = \sin x$ and $R = \sec^2 x$ and

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1 \neq 0.$$

Then, the particular integral is

$$\text{P.I.} = uf(x) + vg(x)$$

where

$$f(x) = -\int \frac{vR}{W} dx = -\int \sin x \sec^2 x dx = -\int \sec x \tan x dx = -\sec x$$

and

$$g(x) = \int \frac{uR}{W} dx = \int \cos x \sec^2 x dx = \int \sec x dx = \ln[\sec x + \tan x]$$

The required solution is

$$y = (c_1 \cos x + c_2 \sin x) - \cos x \ \sec x + \sin x \ln[\sec x + \tan x].$$

Suggested Readings

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