Lesson 7

Linear System of Algebraic Equations – Jacobi Method

7.1 Introduction

In this lesson we discuss an iterative method which gives the solution of a system linear algebraic equations.

Consider a system of algebraic equations as:

$$
\begin{aligned}
&a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\
&a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \\
&\ldots \quad \ldots \quad \ldots \quad \ldots \\
&a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n = b_n
\end{aligned}
\tag{7.1}
$$

This is a linear system of n-algebraic equations in n unknowns. In general, the number of equations is not equal to the number of unknowns. In (7.1), $a_{ij}, i = 1, 2, \ldots, n$; $j = 1, 2, \ldots, n$ are the given coefficients, $b_i, i = 1, 2, \ldots, n$ are the known numbers and $x_i, i = 1, 2, \ldots, n$ are the unknown numbers to be determined.

The system given in (7.1) can also be written in the matrix vector form as:

$$A\underline{x} = \underline{b} \tag{7.2}$$

where $\underline{A} = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix}$, $\underline{b} = \begin{bmatrix} b_1 \\ b_2 \\ \ldots \\ b_n \end{bmatrix}$, $\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ \ldots \\ x_n \end{bmatrix}$

A is called the coefficient matrix.

The solution of this system can be obtained using direct methods such as Gauss Elimination and also by finding the inverse of the matrix A directly. When the size of the matrix A is very large, applicability of these direct methods is limited as finding the inverse of a large size matrix is not so easy. Also, Gauss elimination demands the diagonal dominant structure for A. These difficulties are overcome in the other class of methods called the iterative methods. In what follows, we discuss two useful iterative methods namely the Jacobi and Gauss-Seidel methods.

The iterative methods are based on the idea of successive approximations. Initially, the system of equations is written as:

$$\underline{X}_{n+1} = H\underline{X}_n + \underline{C}\,, n = 0,1,2,... \tag{7.3}$$

where H is the Iterative matrix which depends on the matrix A and $\underline{C}$ is a column vector which depends on both A and $\underline{b}$. We start with an initial approximation to the solution vector $\underline{x} = \underline{x}_0$ and obtain a sequence of approximation to $\underline{x}$ as $\underline{x}_1, \underline{x}_2, ..., \underline{x}_n, ...$, this sequence, in the limit as $n \to \infty$, converge to the exact solution vector $\underline{x}$. We stop the iteration process when the magnitude of the two successive iterates of $\underline{x}$ i.e., $\underline{x}_{n+1}$ and $\underline{x}_n$ is smaller than the pre-assigned error tolerance $\in$, $|\underline{x}_{n+1} - \underline{x}_n| \le \in$ for all elements of $\underline{x}$.

The procedure of obtaining the iterative matrix H is given below.

Let the diagonal elements $a_{ii},\ i=1,2,...,n$ in the linear system (1) do not vanish. We now rewrite the system (1) as:

$$x_1 = \frac{b_1}{a_{11}} - \frac{a_{12}}{a_{11}} x_2 - \frac{a_{13}}{a_{11}} x_3 - ... - \frac{a_{1n}}{a_{11}} x_n$$

$$x_2 = \frac{b_2}{a_{22}} - \frac{a_{21}}{a_{22}} x_1 - \frac{a_{23}}{a_{22}} x_3 - ... - \frac{a_{2n}}{a_{22}} x_n$$

$$\dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots$$

$$x_n = \frac{b_n}{a_{nn}} - \frac{a_{n1}}{a_{nn}} x_1 - \frac{a_{n3}}{a_{22}} x_2 - ... - \frac{a_{n\cdot n-1}}{a_{nn}} x_{n-1} \tag{7.4}$$

The system of equations (7.4) is meaningful only if all a_{ii} (diagonal elements) are non-zero. If some of the diagonal elements in the system of equations given in (7.1) are zero, than the equations should be rearranged so that this condition satisfied. We now form an iterative mechanism for the equation (7.4) by writing

$$x_1^{(n+1)} = \frac{b_1}{a_{11}} - \frac{a_{12}}{a_{11}} x_2^{(n)} - \frac{a_{13}}{a_{11}} x_3^{(n)} - ... - \frac{a_{1n}}{a_{11}} x_n^{(n)}$$

$$x_2^{(n+1)} = \frac{b_2}{a_{22}} - \frac{a_{21}}{a_{22}} x_1^{(n)} - \frac{a_{23}}{a_{22}} x_3^{(n)} - ... - \frac{a_{2n}}{a_{22}} x_n^{(n)}$$

$$\dots \quad \dots \quad \dots \quad \dots$$

$$x_n^{(n+1)} = \frac{b_n}{a_{nn}} - \frac{a_{n1}}{a_{nn}} x_1^{(n)} - \frac{a_{n2}}{a_{nn}} x_2^{(n)} - ... - \frac{a_{n\cdot n-1}}{a_{nn}} x_{n-1}^{(n)}. \tag{7.5}$$

Choose the set of initial approximations as: $\underline{x}^{(0)} = \left(x_1^{(0)}, x_2^{(0)}, ..., x_n^{(0)}\right)$ and generate a sequence of iterates $\underline{x}^{(1)}, \underline{x}^{(2)}, ..., \underline{x}^{(k)}, \underline{x}^{(k+1)}, ...,$ until the convergence condition $\left|\underline{x}^{(k+1)} - \underline{x}^{(k)}\right| \le \in$ is satisfied.

Equation (7.5) is written in the matrix vector form as $\underline{X}^{(n+1)} = H\underline{X}^{(n)} + \underline{C}$

where $H = \begin{bmatrix} 0 & -\frac{a_{12}}{a_{11}} & \dots & -\frac{a_{1n}}{a_{11}} \\ -\frac{a_{21}}{a_{22}} & 0 & \dots & -\frac{a_{2n}}{a_{11}} \\ \dots & \dots & \dots & \dots \\ -\frac{a_{n1}}{a_{nn}} & -\frac{a_{n1}}{a_{nn}} & \dots & 0 \end{bmatrix}$, $C = \begin{bmatrix} \frac{b_1}{a_{11}} \\ \frac{b_2}{a_{22}} \\ \dots \\ \frac{b_n}{a_{nn}} \end{bmatrix}$.

In matrix vector form, the Jacobi method is derived as follows:

Given $A\underline{x} = \underline{b}$, decompose A as the sum of lower Triangular, Diagonal and upper triangular matrices. This decomposition is always possible.

i.e., $A = L + D + U$

$$Ax = (L + D + U)\underline{x} = b$$

or $D\underline{x} = -(L + U)\underline{x} + \underline{b}$

or $\underline{x} = -D^{-1}(L + U)\underline{x} + D^{-1}\underline{b}$

or $\underline{x}^{k+1} = -D^{-1}(L + U)\underline{x}^{k} + D^{-1}b$

or $\underline{x}^{k+1} = H\underline{x}^{k} + \underline{C}$

where the Iteration matrix H in Jacobi method is $-D^{-1}(L + U)$ and C is $D^{-1}b$.

Example 1: Solve the following system of equations using Jacobi method

$$4x_1 + x_2 + x_3 = 2$$

$$x_1 + 5x_2 + 2x_3 = -6$$

$$x_1 + 2x_2 + 3x_3 = -4$$

by taking the initial approximation as $\underline{x}^{(0)} = [0.5, -0.5, -0.5]^T$.

Solution:

Given $A = \begin{bmatrix} 4 & 1 & 1 \\ 1 & 5 & 2 \\ 1 & 2 & 3 \end{bmatrix}$.

Note that $a_{ii} \neq 0$ for $i = 1,2,3$.

First write A as $A = L + D + U$, where

$$D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \; L = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 2 & 0 \end{bmatrix} \text{and } U \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

The iteration matrix H is

$$H = -D^{-1}(L+U) = -\begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 2 & 0 \end{bmatrix}$$

$$= -\begin{bmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{5} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -\frac{1}{4} & \frac{1}{4} \\ -\frac{1}{5} & 0 & -\frac{2}{5} \\ -\frac{1}{3} & -\frac{2}{3} & 0 \end{bmatrix}$$

and the column vector

$$C = D^{-1}b = \begin{bmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{5} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 2 \\ -6 \\ -4 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ -\frac{6}{5} \\ -\frac{4}{3} \end{bmatrix}.$$

We now write the **Jacobi iterative method** as:

$$\underline{x}^{(n+1)} = \begin{pmatrix} x_1^{(n+1)} \\ x_2^{(n+1)} \\ x_3^{(n+1)} \end{pmatrix} = \begin{bmatrix} 0 & -\frac{1}{4} & \frac{1}{4} \\ -\frac{1}{5} & 0 & -\frac{2}{5} \\ -\frac{1}{3} & -\frac{2}{3} & 0 \end{bmatrix} \begin{bmatrix} x_1^{(n)} \\ x_2^{(n)} \\ x_3^{(n)} \end{bmatrix} + \begin{bmatrix} \frac{1}{2} \\ -\frac{6}{5} \\ -\frac{4}{3} \end{bmatrix}, n = 0,1,2,... \qquad \text{..... (i)}$$

Start with the given initial approximation $\begin{bmatrix} x_1^{(0)} \\ x_2^{(0)} \\ x_3^{(0)} \end{bmatrix} = \begin{bmatrix} 0.5 \\ -0.5 \\ -0.5 \end{bmatrix}$, we generate from (i)

$$\begin{bmatrix} x_1^{(1)} \\ x_2^{(1)} \\ x_3^{(1)} \end{bmatrix} = \begin{bmatrix} 0.75 \\ -1.1 \\ -1.1667 \end{bmatrix}.$$

Using this, we generate the next approximation as $\begin{bmatrix} x_1^{(2)} \\ x_2^{(2)} \\ x_3^{(2)} \end{bmatrix} = \begin{bmatrix} 1.0667 \\ -0.8833 \\ -0.85 \end{bmatrix}$, using this, $\begin{bmatrix} x_1^{(3)} \\ x_2^{(3)} \\ x_3^{(3)} \end{bmatrix} = \begin{bmatrix} 0.9933 \\ -1.0733 \\ -1.1 \end{bmatrix}$, ...

The exact solution of this system is $x_1 = 1, x_2 = -1, x_3 = -1$.

Example 2: Solve the following system of linear algebraic equations using the Jacobi method by writing the iterative system directly:
$20x + y - 2z = 17$, $3x + 20 - z = -18$, $2x - 3y + 20z = 25$.

Solution:

We can directly write the iterative system as

$$x_1^{(n+1)} = \frac{1}{20}\left(17 - y^{(n)} + 2z^{(n)}\right)$$

$$y_1^{(n+1)} = \frac{1}{20}\left(-18 - x^{(n)} + z^{(n)}\right)$$

$$z_1^{(n+1)} = \frac{1}{20}\left(25 - 2x^{(n)} + 3y^{(n)}\right) n = 0,1,2,\ldots$$

Start with $x^{(0)} = y^{(0)} = z^{(0)} = 0$.

$$x^{(1)} = 0.85, y^{(1)} = -0.9, z^{(1)} = 1.25.$$

Using this, we generate

$$x^{(2)} = 1.02, y^{(2)} = -0.965, z^{(2)} = 1.1515.$$

In the same way, we generate few more successive approximations as

$$x^{(3)} = 1.0134, y^{(3)} = -0.9954, z^{(3)} = 1.0032$$

$$x^{(4)} = 1.0009, y^{(4)} = -1.0018, z^{(4)} = 0.9993$$

$$x^{(5)} = 1.0, y^{(5)} = -1.0002, z^{(5)} = 0.9996$$

$$x^{(6)} = 1.0, y^{(6)} = -1.0, z^{(6)} = 1.0$$

Thus the solution is: $x = 1.0$, $y = -1.0$, $z = 1.0$.

This is an alternative way of writing the iteration procedure used in the earlier example.

Exercises:

1. Solve the equations

   $10x_1 - 2x_2 - x_3 - x_4 = 3, -2x_1 + 10x_2 - x_3 - x_4 = 15, -x_1 - x_2 + 10x_3 - 2x_4 = 27$
   $-x_1 - x_2 - 2x_3 + 10x_4 = -9$ using the Jacobi method by taking $x_1^{(0)} = 0.3, x_2^{(0)} = x_3^{(0)} = x_4^{(0)} = 0$.

2. Write L, D, U, H and C for the following system of equations: $54x + y + z = 110, 2x + 15y + 6z = 72, -x + 6y + 27z = 85$. Solve this system using the Jacobi iterative method by taking $x^{(0)} = 2, y^{(0)} = 0, z^{(0)} = 0$.

Keywords: System of Linear Algebraic Equations, Jacobi Iterative Method

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Suggested Reading

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