Lesson 8

Gauss-Seidel Iteration Method

8.1 Introduction

This method is applied to the linear system of algebraic equation $Ax = b$ for which the diagonal elements of A are larger in absolute value than the sum of other elements in the each of its row in.

One should arrange, by row and column interchange that larger elements fall along the diagonal, to the maximum possible extent. This method may be seen as an improvement to the Jacobi method, where the available values for the unknowns in a particular iteration are used in the same iteration. Consider the system of linear algebraic equations (5) as given in the lesson 7.

$$x_1^{(1)} = \frac{b_1}{a_{11}} - \frac{a_{12}}{a_{11}} x_2^{(0)} - \frac{a_{13}}{a_{11}} x_3^{(0)} - \ldots - \frac{a_{1n}}{a_{11}} x_n^{(0)}$$

$$x_2^{(1)} = \frac{b_2}{a_{22}} - \frac{a_{21}}{a_{22}} x_1^{(0)} - \frac{a_{23}}{a_{22}} x_3^{(0)} - \ldots - \frac{a_{2n}}{a_{22}} x_n^{(0)}$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots$$

$$x_n^{(1)} = \frac{b_n}{a_{nn}} - \frac{a_{n1}}{a_{nn}} x_1^{(0)} - \frac{a_{n2}}{a_{nn}} x_2^{(0)} - \ldots - \frac{a_{n \cdot n-1}}{a_{nn}} x_{n-1}^{(0)}.$$

This is the first step of the Jacobi iteration method.

In the first step of the iteration, we make use of the initial approximations $x_2^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)}$ in the first of the above equation to evaluate $x_1^{(1)}$ using,

$x_1^{(1)} = \frac{b_1}{a_{11}} - \frac{1}{a_{11}}\left(a_{12}x_2^{(0)} + a_{13}x_3^{(0)} + \ldots + a_{1n}x_n^{(0)}\right)$. This approximation for $x_1^{(1)}$ is used in approximating $x_2^{(1)}$ as shown below:

$$x_2^{(1)} = \frac{b_2}{a_{22}} - \frac{1}{a_{22}}\left(a_{21}x_2^{(1)} + a_{23}x_3^{(0)} + \ldots + a_{2n}x_n^{(0)}\right)$$

Likewise, the other unknown are found as shown below:

$$x_3^{(1)} = \frac{b_3}{a_{33}} - \frac{1}{a_{33}}\left(a_{31}x_1^{(1)} + a_{32}x_2^{(1)} + \ldots + a_{3n}x_n^{(0)}\right)$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots$$

$$x_n^{(1)} = \frac{b_n}{a_{nn}} - \frac{1}{a_{nn}}\left(a_{n1}x_1^{(1)} + a_{n2}x_2^{(1)} + \ldots + a_{n\cdot n-1}x_n^{(1)}\right)$$

We proceed to find the second approximation for the solution in the same manner. The iteration process is terminated using the same criterion that was discussed in the case of Jacobi method.

We express the Gauss-Seidel method in the matrix form as follows for $Ax = b$. Let $A = L + D + U$, where L and U are the lower and upper triangular matrices with zeros for the diagonal entries and D is the diagonal matrix.

Then, $Ax = (L + D + U)x = b$.

$$(L + D)\underline{x}^{(k+1)} = b - U\underline{x}^{(k)}$$

$$\underline{x}^{(k+1)} = (D + L)^{-1}b - (D + L)^{-1}U\underline{x}^{(k)}$$

or $\underline{x}^{(k+1)} = H\underline{x}^{(k)} + C$

where $H = -(D+L)^{-1} \cdot U$ and $C = (D+L)^{-1} b$.

Let us illustrate the use of Gauss-Seidel method below.

Example 1: Solve the system of equations using the Gauss-Seidel method correct to three decimal places:

$x + 2y + z = 0$ ..... (i)

$3x + y - z = 0$ ..... (ii)

$x - y + 4z = 3$ ..... (iii)

Solution:

In the above equations, note that

i. $|1| > |2| + |1|$, i.e., $|1| > |3|$ is false.

ii. $|1| > |3| + |-1|$ i.e., $1 > 4$ is false.

iii. $|4| > |1| + |-1|$ i.e., $4 > 2$ is true.

Thus in the first two equations, the diagonal dominance is not present.
We now rearrange the order of the given system as:

$3x + y - z = 0$

$x + 2y + z = 0$

$x - y + 4z = 3$

Now this system shows diagonal dominance as $3 \geq 2, 2 \geq 2, 4 > 2$.

We now write the iterative process for the above as:

$$x^{(k+1)} = \frac{1}{3}\left[-y^{(k)} + z^{(k)}\right]$$

$$y^{(k+1)} = \frac{1}{2}\left[-x^{(k+1)} - z^{(k)}\right]$$

$$z^{(k+1)} = \frac{1}{4}\left[3 - x^{(k+1)} + y^{(k+1)}\right],\ k = 0,1,2,...$$

We start with the initial guess values as $x^{(0)} = 1, y^{(0)} = 1, z^{(0)} = 1$.

This gives,

$$x^{(1)} = \frac{1}{3}\left[z^{(0)} - y^{(0)}\right] = 0$$

$$y^{(1)} = \frac{1}{2}\left[-x^{(1)} - z^{(0)}\right] = -0.5$$

$$z^{(1)} = \frac{1}{4}\left[3 - x^{(1)} + y^{(1)}\right] = 0.625.$$

The second iteration is given by

$$x^{(2)} = \frac{1}{3}\left[z^{(1)} - y^{(1)}\right] = 0.375$$

$$y^{(2)} = \frac{1}{2}\left[-x^{(2)} - z^{(1)}\right] = -0.5$$

$$z^{(2)} = \frac{1}{4}\left[3 - x^{(2)} + y^{(2)}\right] = 0.53125.$$

Proceeding in this way, we generate the sequence of approximations as:

$$x^{(3)} = 0.34375, y^{(3)} = -0.4375, z^{(3)} = 0.55469$$

$$x^{(4)} = 0.33075, y^{(4)} = -0.44271, z^{(4)} = 0.55664$$

$$x^{(5)} = 0.33312, y^{(5)} = -0.4449, z^{(5)} = 0.5555.$$

The approximate solution correct to 3 decimal places is take as:

$x = 0.333, y = -0.445, z = 0.555.$

Example 2: Solve the system of equations $2x - y = 7, -x + 2y - z = 1, -y + 2z = 1$ using the Gauss-Seidel method by taking the initial approximations as $x^{(0)} = 0, y^{(0)} = 0, z^{(0)} = 0$.

Solution:

Given $A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$, $b = \begin{pmatrix} 7 \\ 1 \\ 1 \end{pmatrix}$, we find the decomposition of A as

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}, L = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}, U = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Gauss-Seidel method is $\underline{x}^{(k+1)} = -(D+L)^{-1} \cdot U\underline{x}^{(k)} + (D+L)^{-1}b$.

$$(D+L) = \begin{pmatrix} 2 & 0 & 0 \\ -1 & 2 & 0 \\ 0 & -1 & 2 \end{pmatrix}, (D+L)^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ \frac{1}{4} & \frac{1}{2} & 0 \\ \frac{1}{8} & \frac{1}{4} & \frac{1}{2} \end{bmatrix},$$

$$(D+L)^{-1}U = \begin{bmatrix} 0 & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{4} & -\frac{1}{2} \\ 0 & -\frac{1}{8} & -\frac{1}{4} \end{bmatrix}, \quad (D+L)^{-1}b = \begin{bmatrix} \frac{7}{2} \\ \frac{9}{4} \\ \frac{13}{8} \end{bmatrix}.$$

Applying the above method,

$$\begin{pmatrix} x^{(1)} \\ y^{(1)} \\ z^{(1)} \end{pmatrix} = \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{4} & \frac{1}{2} \\ 0 & \frac{1}{8} & \frac{1}{4} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{7}{2} \\ \frac{9}{4} \\ \frac{13}{8} \end{bmatrix} = \begin{bmatrix} 3.5 \\ 2.25 \\ 1.625 \end{bmatrix}$$

Using $\begin{pmatrix} x^{(1)} \\ y^{(1)} \\ z^{(1)} \end{pmatrix}$, we generate $\begin{pmatrix} x^{(2)} \\ y^{(2)} \\ z^{(2)} \end{pmatrix} = \begin{pmatrix} 4.625 \\ 3.625 \\ 2.3125 \end{pmatrix}$,

Similarly, $\begin{pmatrix} x^{(3)} \\ y^{(3)} \\ z^{(3)} \end{pmatrix} = \begin{pmatrix} 5.3125 \\ 4.3125 \\ 2.6563 \end{pmatrix}$.

Note that these iterates are approaching the exact solution $x = 6,\ y = 5,\ z = 3$.

Exercises: By choosing suitable initial approximations, solve the following system of linear algebraic equations using Gauss-Seidel method.

1. $8x + 2y - 2z = 8,\ x - 8y + 3z = -4,\ 2x + y + 9z = 12.$
2. $2x + 10y + z = 51,\ x - 2y + 8z = 5,\ 15x + 3y - 2z = 85.$
3. $x + 2y + z = 0,\ 3x + y - 5 = 0,\ x - y + 4z = 3.$

4. $10x_1 - 2x_2 - x_3 - x_4 = 3, -2x_1 + 10x_2 - x_3 - x_4 = 15, -x_1 - x_2 + 10x_3 - 2x_4 = 27,$
   $-x_1 - x_2 - 2x_3 + 10x_4 = -9$.

Keywords: Linear Algebraic Equations, Gauss-Seidel Method, Diagonal Dominance

References

Jain, M. K., Iyengar. S.R.K., Jain. R.K. (2008). Numerical Methods. Fifth Edition, New Age International Publishers, New Delhi.

Atkinson, E Kendall. (2004). Numerical Analysis. Second Edition, John Wiley & Sons, Publishers, Singapore.

Suggested Reading

Scheid, Francis. (1989). Numerical Analysis. Second Edition, McGraw-Hill Publishers, New York.

Sastry, S.S. (2005). Introductory Methods of Numerical Analysis. Fourth Edition, Prentice Hall of India Publishers, New Delhi.