Lesson 17

Newton's Divided Difference Interpolation

17.1 Introduction

In Lagrange interpolation, the fundamental polynomials are constructed for writing the interpolating polynomial. Suppose we found the fundamental polynomials for the given N data point set. If a data point is added to this set, then the fundamental polynomials are reconstructed. This makes the process laborious. An easier way of finding an interpolating polynomial is given by constructing the divided differences.

17.2 Divided Difference

For the set $\{(x_0, f_0),(x_1, f_1)\}$, the linear interpolating polynomial $a_0 + a_1 x$ is given by:

$$\begin{vmatrix} p(x) & x & 1 \\ f(x_0) & x_0 & 1 \\ f(x_1) & x_1 & 1 \end{vmatrix} = 0 .$$

Expand the determinant in term of the first row, we get

$$p(x)(x_0 - x_1) - x[f(x_0) - f(x_1)] + 1[x_1 f(x_0) - x_0 f(x_1)] = 0$$

or it is rewritten as

$$\begin{aligned} p(x) &= f(x_0) + (x - x_0)\frac{f(x_1) - f(x_0)}{x_1 - x_0} \\ &= f(x_0) + (x - x_0) f[x_0, x_1] \qquad (17.1) \end{aligned}$$

where $f[x_0, x_1]$ is defined as the first divided difference of $f(x)$ relative to x_0 and x_1, given as:

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}.$$

Example 1: Given $f(2) = 4, f(2.5) = 5.5$, find the linear interpolating polynomial using Newton's divided difference interpolation.

Solution:

Given $x_0 = 2,\ f_0 = 4,\ x_1 = 2.5,\ f_1 = 5.5.$

Newton's first divided difference

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{5.5 - 4}{2.5 - 2} = \frac{1.5}{0.5} = 3.$$

The interpolating polynomial is

$$p_1(x) = f(x_0) + (x - x_0) f[x_0, x_1]$$

$$= 4 + (x - 2)(3)$$

$$= 3x - 2.$$

17.3 Generalization to $(N + 1)$ Data Points

The second divided difference of $f(x)$ relative to the points x_0, x_1, x_2 is written as:

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{(x_2 - x_0)}$$

where $f[x_1, x_2]$ is the first divided difference of $f(x)$ relative to x_1 and x_2 is given by:

$$f[x_1, x_2] = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

The third divided difference of $f(x)$ relative to x_0, x_1, x_2, x_3 is given by

$$f[x_0, x_1, x_2, x_3] = \frac{f[x_1, x_2, x_3] - f[x_0, x_1, x_2]}{(x_3 - x_0)}.$$

The same way, the nth divided difference is written as

$$f[x_0, x_1, ..., x_N] = \frac{f[x_1, x_2, ..., x_N] - f[x_0, x_1, x_2, ..., x_{N-1}]}{(x_N - x_0)}.$$

Let $p(x) = a_0 + (x - x_0)a_1 + ... + (x - x_0)(x - x_1)...(x - x_{N-1})a_N$ be the interpolating polynomial for the $(N+1)$ distinct points $\{(x_0, f_0), (x_1, f_1), ..., (x_N, f_N)\}$.

Substituting $x = x_0$ in the above, we get

$$p(x_0) = a_0 = f(x_0) = f[x_0].$$

Put $x = x_1$, we obtain

$p(x_1) = a_0 + (x - x_0)a_1 = f[x_1]$ (say)

$$\Rightarrow a_1 = \frac{f(x_1) - f(x_0)}{(x_1 - x_0)} = f[x_0, x_1].$$

Put $x = x_2$, we get

$$p(x_2) = f(x_0) + (x - x_0) f[x_0, x_1] + (x_2 - x_0)(x_2 - x_1) a_2 = f(x_2).$$

On simplifying, we get

$$a_2 = f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{(x_2 - x_0)}.$$

Proceeding in this way, we show that

$$a_n = f[x_0, x_1, ..., x_N].$$

Then we obtain the divided difference interpolating polynomial as

$$p_N(x) = f(x_0) + (x - x_0) f[x_0, x_1] + ... + (x - x_0)(x - x_1)...(x - x_{N-1}) f[x_0, x_1, ..., x_N]. \tag{17.2}$$

This formula is easily extended to (say) $(N + 2)$ data. (i.e., addition of one more data point to the previous data set) as

$$p_{N+1}(x) = f(x_0) + (x - x_0) f[x_0, x_1] + ... + (x - x_0)...(x - x_{N-1}) f[x_0, x_1, ..., x_N] + (x - x_0)...(x - x_N) f[x_0, x_1, ..., x_N, x_{N+1}] \tag{17.3}$$

It amounts to finding the next divided difference and adding it to previously obtained interpolating polynomial as shown above.

Example 2: Find the Newton divided difference interpolating polynomial for the data.

x:	0	1	3
f:	1	3	5.5

Solution:

$x_0 = 0, x_1 = 1, x_2 = 3, f_0 = 1, f_1 = 3, f_2 = 55$.

$$f[0,1] = \frac{f(1) - f(0)}{1-0} = \frac{3-1}{1} = 2,$$

$$f[1,3] = \frac{55-3}{3-1} = \frac{52}{2} = 26,$$

$$\therefore f[0,1,3] = \frac{26-2}{3-0} = \frac{24}{3} = 8.$$

$\therefore$ Newton's divided difference interpolating polynomial is

$$p_2(x) = f(0) + (x-0)f[0,1] + (x-0)(x-1)f[0,1,3]$$

$$= 1 + x \cdot 2 + (x)(x-1)8$$

$$= 8x^2 - 6x + 1.$$

Exercises:

1. Find $f(7)$ from the following data using the Newton's divided difference interpolation.

x:	1.5	3.0	5.0	6.5	8.0
f:	5.0	31.0	131.0	282.0	521.0

2. If $f(x) = \frac{1}{x^2}$, find the divided difference $f[x_0, x_1, x_2]$.

3. From the data

x:	-1	1	4	7
$f(x)$:	-2	0	63	342

(i) Find the Lagrange interpolating polynomial.

(ii) Find the Newton divided difference interpolating polynomial.

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Suggested Reading

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