Lesson 20

Simpson's one Third and Simpson's Three Eighth Rules

20.1 Simpson's One-Third Rule

It is obtained by taking $N = 2$ in the Newton-Cotes formula (4). We described this in case (2) of the lesson (19). We divide the interval $[a,b]$ into an even number of subintervals of equal length having odd number of abscissas.

We divide the interval $[a,b]$ into $2k$ subintervals each of length $h = \frac{b-a}{2k}$, we then get $2k+1$ abscissas as

$$a = x_0 < x_0 < x_0 < \cdots < x_{2k} = b, \qquad x_i = x_0 + ih, \qquad i = 1,2,\ldots,2k-1.$$

Now $I = \int_a^b y(x)dx$

$$= \int_{x_0}^{x_2} y(x)dx + \int_{x_2}^{x_4} y(x)dx + \cdots + \int_{x_{2k-2}}^{x_{2k}} y(x)dx.$$

Now using the Newton-Cotes formula with $N = 2$ for each of the above integrals, we get

$$\int_{x_0}^{x_2} y(x)dx = \frac{h}{3}[y_0 + 4y_1 + y_2]$$

$$\int_{x_2}^{x_4} y(x)dx = \frac{h}{3}[y_2 + 4y_3 + y_4]$$

...

$$\int_{x_{2k-2}}^{x_{2k}} y(x)dx = \frac{h}{3}[y_{2k-2} + 4y_{2k-1} + y_{2k}]$$

Add all these values, we obtain

$$I = \int_a^b y(x)dx$$

$$= \frac{h}{3}[y_0 + 4(y_1 + y_3 + \cdots + y_{2k-1}) + 2(y_2 + y_4 + \cdots + y_{2k-2}) + y_{2k}]$$

or,

$$I = \frac{h}{3}[X + 4.0 + 2E] \qquad (20.1)$$

where X = sum of the function values at the end points, O= sum of the function values at odd numbered abscissas, and E = sum of the function values at even numbered abscissas.

This formula is known as the Simpson's $\frac{1}{3}$rd rule of integration.

Example: Evaluate $\int_0^{12} \frac{1}{1+x^2} dx$ by Simpson's $\frac{1}{3}$rd rule taking $k = 6$.

Solution:

$y(x) = \frac{1}{1+x^2}$, $a = 0, b = 12, k = 6, h = 2$

$y(x_i) = y_i; i = 0,1,2,3,4,5.$

x:	0	2	4	6	8	10	12
$y(x)$:	1	0.2	0.05882	0.02703	0.01538	0.0099	0.0069

By Simpson's $\frac{1}{3}$rd rule:

$$I = \frac{h}{3}[(y_0 + y_6) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)]$$

$$= \frac{2}{3}[(1 + 0.0069) + 4(0.2 + 0.02703 + 0.0099) + 2(0.05882 + 0.01538)]$$

$$= 1.40201.$$

20.2 Simpson's Three-Eighth Rule

Take $N = 3$ in the Newton-Cotes formula (4). This is described in Case (3) of the lesson (19).To apply this method the number of subintervals should be taken as multiples of 3.

The integral

$$I = \int_{x_0}^{x_N = x_0 + Nh} y(x)dx$$

$$= \int_{x_0}^{x_0+3h} y(x)dx + \int_{x_0+3h}^{x_0+6h} y(x)dx + \ldots + \int_{x_{N-3}}^{x_N} y(x)dx \tag{20.2}$$

This is known as the Simpson's $\frac{3}{8}$th rule.

Example 2: Evaluate $\int_0^6 \frac{1}{1+x^2} dx$ by Simpson's $\frac{3}{8}$th rule.

Solution:

Take $h = 1, x_0 = 0, x_6 = 6$, $f(x) = \dfrac{1}{1+x^2}$.

The number of subintervals is 6, is a multiple of 3. So we can use the Simpson's $\frac{3}{8}$th rule.

x:	0	1	2	3	4	5	6
y:	1 y_0	0.5 y_1	0.2 y_2	0.1 y_3	0.0588 y_4	0.0385 y_5	0.027 y_6

$$I = \frac{3h}{8}\left[(y_0 + y_6) + 3(y_1 + y_2 + y_4 + y_5) + 2(y_3)\right] = 1.3571.$$

Example 3: Using Simpson's $\frac{1}{3}$rd rule, find $\int_0^{0.6} e^{-x^2}dx$ by taking 6 subintervals.

Solution:

Evaluation of e^{-x^2} is not a simple function, that cannot be integrated directly. In such a situation using numerical integration it can be easily evaluated. Let us construct the data:

x:	x_0 0	x_1 0.1	x_2 0.2	x_3 0.3	x_4 0.4	x_5 0.5	x_6 0.6
x^2:	0	0.01	0.04	0.09	0.16	0.25	0.36
$y = e^{-x^2}$	1 y_0	0.99 y_1	0.9608 y_2	0.9139 y_3	0.8521 y_4	0.7788 y_5	0.6977 y_6

By Simpson's $\frac{1}{3}$rd rule, we have

$$\int_0^{0.6} e^{-x^2}dx = \frac{h}{3}\left[(y_0 + y_6) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)\right]$$

$$= \frac{0.1}{3}\left[(1 + 0.6977) + 4(0.99 + 0.9139 + 0.8521) + 2(0.9608 + 0.8521)\right]$$

$$= \frac{0.1}{3}\left[1.6977 + 10.7308 + 3.6258\right]$$

$$= 0.5351.$$

Example 4: A solid of revolution is formed by rotating about the x-axis the area bounded by the x-axis, the lines $x = 0$ and $x = 1$, and the curve through the point with the following data:

x:	0	0.25	0.5	0.75	1.0
y:	1.0	0.9896	0.9589	0.9089	0.8415

	y_0	y_1	y_2	y_3	y_4

Estimate the volume of the solid using Simpson's $\frac{1}{3}$rd rule.

Solution:

Here $h = 0.25$.

The volume of the solid of revolution is

$$I = \int_0^1 \pi y^2 dx$$

Using the Simpson's $\frac{1}{3}$rd rule:

$$I = \frac{h}{3}\pi\left[\left(y_0{}^2 + y_4{}^2\right) + 4\left(y_1{}^2 + y_3{}^2\right) + 2\left(y_2{}^2\right)\right]$$

$$= \frac{0.25}{3}\frac{22}{7}\left[\left\{(1)^2 + (0.8415)^2\right\} + 4\left\{(0.9896)^2 + (0.9089)^2\right\} + 2(0.9589)^2\right]$$

$$= 0.2618[10.7687]$$

$$= 2.8192.$$

Exercises:

1. Evaluate $\int_4^{5.2} \log x \cdot dx$

Using (i) Trapezoidal rule

(ii) Simpson's $\frac{1}{3}$rd rule

(iii) Simpson's $\frac{3}{8}$th rule

by taking 12 subintervals. Then compare your results.

2. A curve is given by

x	0	1	2	3	4	5	6
y	0	2	2.5	2.3	2	1.7	1.5

Evaluate (i) the area below the given curve

(ii) $\int_0^6 xy \cdot dx$ using Simpson's $\frac{1}{3}$rd rule.

3. Estimate the length of the arc of the curve $3y = x^3$ from $(0,0)$ to $(1,3)$ using Simpson's $\frac{1}{3}$rd rule by taking 8 subintervals.

4. Evaluate $I = \int_0^{\frac{\pi}{2}} \sqrt{\cos\theta} \cdot d\theta$ by using Simpson's $\frac{1}{3}$rd rule using 11 ordinates.

Keyword: Simpson's one-Third rule, Simpson's three-eighth rule, Even number of subintervals.

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Suggested Reading

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