# Lesson 22 Gaussian Quadrature

**22.1 Introduction:** The problem of numerical integration is to find an approximate value for

$$I = \int_{a}^{b} w(x) f(x) dx \qquad (22.1)$$

where w(x) is a positive valued continuous function defined on [a,b] called the weight function. The function w(x)f(x) is assumed to be integrable. The limits a and b are finite, semi-infinite or infinite. The integral (22.1) is approximated by a finite linear combination of  $f(x_k)$  in the form

$$I = \int_{a}^{b} w(x) f(x) dx = \sum_{k=0}^{N} \lambda_{k} f_{k}$$
(22.2)

where  $x_k, k = 0, 1, ..., N$  are called the nodes which are distributed within the limits of integration **[a, b]** and  $\lambda_k, k = 0, 1, ..., N$  are called the discrete weights. The formula (**22.2**) is also known as the quadrature formula.

The error in this approximation is given as

$$R_{N} = \int_{a}^{b} w(x) f(x) \, dx - \sum_{k=0}^{N} \lambda_{k} f_{k}$$
(22.3)

An integration method of the form (22.2) is said to be order p if it produces exact results i.e.;  $R_N = 0$  for all polynomials of degree less than or equal to p. In evaluating the integral (22.1) using (22.2) involves finding (N + 1) unknown weights  $\lambda_k$ 's and (N + 1) unknown nodes  $x_k$ 's leading to computing (2N+2) unknowns. To compute these unknowns, the method (22.2) is made exact for polynomial of degree less than or equal to (2N+1), for example, by considering  $f(x) = c_0 + c_1 x + c_2 x^2 + ... + c_{2N+1} x^{2N+1}$ .

For example, when n = 2, then  $\int_{-1}^{1} f(x) dx = w_1 f(x_1) + w_2 f(x_2)$ .

When the nodes  $x_k$  are known, the corresponding methods are called Newton-Cotes methods where the nodes are also to be determined, then the methods are called the quadrature methods.

The interval of integration **[a,b]** is always transformed to **[-1,1]** using the transformation  $x = \left(\frac{b-a}{2}\right)t + \left(\frac{b+a}{2}\right)$ . Depending on the weight function **w(x)** 

a variety of methods are developed. We discuss here the Gauss-Legendre integration method for which the weight function w(x) = 1.

#### 22.2 Gauss-Legendre Integration Methods

Consider evaluating the integral

$$I = \int_{-1}^{1} f(x) dx = \sum_{k=0}^{N} \lambda_k f(x_k)$$

where  $x_k$  are the nodes and  $\lambda_k$  are the weights.

(I) One Point formula n = 0: The formula is  $\int_{-1}^{1} f(x) dx = \lambda_0 f(x_0)$ 

In the above  $\lambda_0, x_0$  are unknowns, these are obtained by making this integration method exact for f(x) = 1, x.

i.e., (a) 
$$\int_{-1}^{1} 1 \cdot dx = \lambda_0 f(x_0) = \lambda_0 = 2$$
  
(b) 
$$\int_{-1}^{1} x \cdot dx = \lambda_0 x_0 \Longrightarrow \lambda_0 x_0 = 0 \Longrightarrow x_0 = 0 (\because \lambda_0 = 2).$$
$$\therefore \int_{-1}^{1} f(x) dx = 2 \cdot f(0).$$

(II) The two point formula n = 1: The formula is given by

$$\int_{-1}^{1} f(x) dx = \lambda_0 f(x_0) + \lambda_1 f(x_1)$$

The unknowns are  $\lambda_0, \lambda_1, x_0, x_1$ . These unknowns are determined by making this method exact for  $f(x) = 1, x, x^2, x^3$ ; we get

$$f(x) = 1 \Longrightarrow \lambda_0 + \lambda_1 = 2.$$
  

$$f(x) = x \Longrightarrow \lambda_0 x_0 + \lambda_1 x_1 = 0.$$
  

$$f(x) = x^2 \Longrightarrow \lambda_0 x_0^2 + \lambda_1 x_1^2 = \frac{2}{3}.$$
  

$$f(x) = x^3 \Longrightarrow \lambda_0 x_0^3 + \lambda_1 x_0^3 = 0.$$

Solving these non-linear equations we obtain

$$x_0 = \pm \frac{1}{\sqrt{3}}, x_1 = \mp \frac{1}{\sqrt{3}}, \ \lambda_0 = \lambda_1 = 1.$$

And the two point Gauss-Legendre method is given by

$$\int_{-1}^{1} f(x) dx = f(-\frac{1}{\sqrt{3}}) + f(\frac{1}{\sqrt{3}}).$$

Exercise: Show that the three point Gauss-Legendre method is given by

$$\int_{-1}^{1} f(x)dx = \frac{1}{9} [5f(-\sqrt{\frac{3}{5}}) + 8f(0) + 5f(\sqrt{\frac{3}{5}})].$$

**Example 1:** Evaluate the integral  $I = \int_{1}^{2} \frac{2x}{1+x^4} dx$ 

using the Gauss-Legendre 2-point quadrature rule.

# Solution:

The general quadrature formula is written in [-1,1].

So define  $x = (\frac{b-a}{2})t + (\frac{b+a}{2})t \implies x = \frac{1}{2}t + \frac{3}{2}, \ dx = \frac{1}{2}dt$ 

The integral transforms to

$$\int_{-1}^{1} \frac{8(t+3)}{\left[16 + (t+3)^{4}\right]} dt.$$

Using then 3-point rule, we get

$$I = \frac{1}{9} \left[ 5f\left(-\sqrt{\frac{3}{5}}\right) + 8f(0) + 5f\left(\sqrt{\frac{3}{5}}\right) \right]$$
$$= \frac{1}{9} \left[ 5\left(0.4393\right) + 8(0.2474) + 5\left(0.1379\right) \right]$$
$$= 0.5406$$

We can directly integrate  $\int_{1}^{2} \frac{2x}{1+x^4} dx$  and its integral is  $\tan^{-1}(4) - \frac{\pi}{4} = 0.5404$ .

**Example 2:** Evaluate the integral  $\int_{0}^{1} \frac{1}{1+x} dx$  using the Gauss-Legendre two point

formula.

#### Solution:

Define 
$$x = \frac{1}{2}t + \frac{1}{2} \Rightarrow dx = \frac{1}{2}dt$$
.  

$$\therefore \int_{-1}^{1} \frac{dt}{t+3} = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$$

$$= 0.69231$$

**Exercises:** Evaluate (a) 
$$\int_{0}^{1} \frac{1}{1+x} dx$$
 (b)  $\int_{1}^{2} \frac{1}{x} dx$  (c)  $\int_{1}^{2} \frac{1}{1+x^{4}} dx$ 

using Gauss-Legendre (i) 2-point (ii) 3-point quadrature methods.

Keywords: Gaussian Quadrature, One Point formula, Two point formula.

## References

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## **Suggested Reading**

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