Lesson 24

Non Homogeneous Difference Equation

24.1 Introduction

In this lesson we learn how to find the solution of the non-homogeneous difference equation. The solution corresponding to the non-homogeneous term is called the Particular integral of the difference equation:

Consider the non-homogeneous difference equation in the form

$$y_{n+k} + \alpha_1 y_{n+k-1} + ... + \alpha_k y_k = f(n) \tag{24.1}$$

Note that equation (24.1) is equivalent to equation (23.6).

Using the shift operator E, the above can be put in the operator form as

$$\varphi(E) y_n = f_n \tag{24.2}$$

where $\varphi(E) = E^k + \alpha_1 E^{k-1} + ... + \alpha_k$.

Then the Particular integral is written as:

$$V_n = \frac{1}{\varphi(E)} f(n)$$

24.2 Finding the Particular Integral:

$\varphi(E)$ is an operator involving E, $\frac{1}{\varphi(E)}$ is its inverse operator [assuming its existence]. The particular solution is obtained for different forms of non-homogeneous function $f(n)$ as given below. We consider the forms for $f(n)$ as a^n, $\sin pn$, $\cos pn$ and $n^p a^n \cdot G(n)$ where $G(n)$ is a polynomial in n.

(A) When $f(n) = a^n$, the particular integral $(P.I.)$ is

$\frac{1}{\varphi(E)}a^n = \frac{1}{\varphi(a)}a^n$ provided $\varphi(a) \neq 0$. If for $\varphi(a) = 0$, a is a simple root,

then $\frac{1}{E-a}a^n = na^{n-1}$ and if 'a' is a root with multiplicity $m\ (m < n)$ then

$(E-a)^m = 0$ and the particular integral is

$$\frac{1}{(E-a)^m} = \frac{n(n-1)....(n-m)}{m!}a^{n-m}.$$

This way one can find the P.I. V_n for the given non-homogeneous equation when $f(n) = a^n$.

Example 1: Find the particular integral of $y_{n+2} - 4y_{n+1} + 3y_n = 2^n + 5^n$.

Solution:

$$P.I. = \frac{1}{(E^2 - 4E + 3)}(2^n + 5^n)$$

$$= \frac{1}{(E^2 - 4E + 3)}2^n + \frac{1}{(E^2 - 4E + 3)}5^n$$

Clearly, 2 and 5 are not the roots of the auxiliary equation of $E^2 - 4E + 3$.

$$P.I. = \frac{1}{2^2 - 4\cdot 2 + 3}2^n + \frac{1}{5^2 - 4\cdot 5 + 3}5^n$$

$$= -2^n + \frac{1}{8}5^n$$

Example 2: Solve the difference equation $y_{n+2} - 6y_{n+1} + 9y_n = 3^n + (-1)^n$.

Solution:

The characteristic equation is $\xi^2 - 6\xi + 9 = 0$.

Roots are $\xi = 3,3$.

The complementary function is: $y_{C.F.} = (\alpha_1 + h\alpha_2)(3^n)$.

Note that 3 is a root of the characteristic equation, with multiplicity 2, the particular integral is written as $\frac{n(n-1)}{2!} \cdot 3^{n-2} + \frac{1}{15}(-1)^n$.

Hence the complete solution is $y_n = y_{C.F} + y_{P.I.} = Y_n + V_n$

$$= (\alpha_1 + \alpha_2 n)3^n + \frac{n(n-1)}{2!} \cdot 3^{n-2} + \frac{1}{15}(-1)^n.$$

(B) When $f(n) = \sin pn$ or $\cos pn$:

$$P.I. = \frac{1}{\varphi(E)} \sin pn = \frac{1}{\varphi(E)} \left(\frac{e^{ipn} - e^{-ipn}}{2i} \right)$$

$$= \frac{1}{2i} \left(\frac{1}{\varphi(E)} a^n - \frac{1}{\varphi(E)} b^n \right); \begin{cases} a^n = e^{ipn} \\ b^n = e^{-ipn} \end{cases}$$

This is in the form discussed for $f(n) = a^n$.

Similarly, $\frac{1}{\varphi(E)} \cos pn = \frac{1}{2} \left(\frac{1}{\varphi(E)} a^n + \frac{1}{\varphi(E)} b^n \right)$.

(C) When $f(n) = n^p$, then

$$P.I. = \frac{1}{\varphi(E)} n^p = [\varphi(E)]^{-1} n^p.$$

Recall $E = 1 + \Delta$.

$\therefore P.I. = \left[\varphi(E)\right]^{-1} n^p$.

Expanding $\left[\varphi(1+\Delta)\right]^{-1}$ in increasing powers of Δ (using binomial theorem) and operating it over n^p, we get the particular integral.

(D) When $f(n) = a^n a(n)$, where $a(n)$ is a polynomial in n.

$$P.I. = \frac{1}{\varphi(E)} a^n a(n) = a^n \frac{1}{\varphi(aE)} a(n).$$

This can be solved using the procedure given in (C).

Example 3: Solve $y_{n+2} - 2\cos\alpha\, y_{n+1} + y_n = \cos n$.

Solution:

It can be readily seen that the characteristic equation as

$\xi^2 - 2\cos\alpha\xi + 1 = 0$ and its roots are $\cos\alpha \pm i\sin\alpha$.

So the C.F. is: $\left(\alpha_1 \cos\alpha n \pm \alpha_2 \sin\alpha n\right)(1)^n$.

$$P.I. = \frac{1}{E^2 - 2E\cos\alpha + 1}\cos\alpha = \frac{1}{E^2 - 2E\left(e^{i\alpha} + e^{-i\alpha}\right) + 1}\frac{\left(e^{in} + e^{-in}\right)}{2}$$

$$= \frac{1}{2}\left[\frac{1}{E - e^{i\alpha}} \cdot \frac{1}{\left(e^{i\alpha} - e^{-i\alpha}\right)} \cdot e^{in} + \frac{1}{E - e^{-i\alpha}} \cdot \frac{1}{\left(e^{-i\alpha} - e^{i\alpha}\right)} \cdot e^{-in}\right]$$

$$= \frac{1}{4i\sin\alpha}\left[\frac{1}{E - e^{i\alpha}} \cdot e^{in} - \frac{1}{E - e^{-i\alpha}} \cdot e^{-in}\right]$$

$$= \frac{1}{4i\sin\alpha}\left[\frac{1}{e^{i} - e^{i\alpha}} \cdot e^{in} - \frac{1}{e^{-i} - e^{-i\alpha}} \cdot e^{-in}\right]$$

$\therefore y_n = C.F. + P.I.$

Example 4: Find the particular integral of $y_{n+2} - 2y_{n+1} + y_n = n \cdot 3^n$.

Solution:

$$P.I. = \frac{1}{(E-1)^2} \cdot n \cdot 3^n$$

$$= 3^n \cdot \frac{1}{(3E-1)^2} \cdot n$$

$$= 3^n \cdot \frac{1}{2^2\left(1+\frac{3}{2}\Delta\right)^2} \cdot n$$

$$= \frac{3^n}{2^2}\left(1+\frac{3}{2}\Delta\right)^{-2} \cdot n$$

$$= \frac{3^n}{2^2}(1-3\Delta+....)[n]$$

$$= \frac{3^n}{2^2}\{[n]-3\cdot 1\} = \frac{3^n}{2^2}(n-3).$$

Exercises:

1. Solve $y_{n+2} - 4y_n = n-1$.

2. Solve the following difference equations:

   (i) $\Delta^2 y_n - 5\Delta y_n + 4y_n = n + 2^n$

   (ii) $y_{n+2} - 6y_{n+1} + 8y_n = 2^n + 6n$

   (iii) $y_{n+2} - \left(2\cos\frac{1}{2}\right)y_{n+1} + y_n = \sin\frac{n}{2}$.

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