Lesson 25

Numerical Solutions of Ordinary Differential Equations

25.1 Introduction

Many problems in engineering and science are modelled as ordinary differential equations which are either linear or non-linear equations satisfying certain given conditions. Only a few of these equations can be solved using the standard analytical methods whereas the other alternative is to find their numerical solution. Here, we see some of the numerical methods to solve a class of problem known as the initial value problems (I.V.P). An initial value problem is one where the differential equation is solved subjected to the required number of initial of initial conditions.

A general first order initial value problem is given by $\frac{dy}{dt} = f(y,t)$ (25.1)

$t \in I = [t_0, b]$; subjected to $y(t_0) = y_0$ (25.2)

The solution of (25.1) - (25.2) can be found as a series for y in terms of power of the independent variable 't', from which the value of y can be obtained by direct substitutionor as a set of tabled values of t and y.

The methods due to Picard and Taylor (Series) find the solution of the IVP (25.1) -(25.2) as the dependent function y as a function of the independent variable 't'. The other methods such as the Euler and Runge-Kutta methods give the solution y at some discrete data set for t in the interval $I = [t_0, b]$.

For an nth order differential equation, the general solution has n arbitrary constants and in order to compute the numerical solution of such an equation,

we need n initial conditions each on the function and its derivatives upto the $(n-1)^{th}$ order at the initial value I.V.P can also be found in the similar manner as that of the solution of (25.1)-(25.2), but by using the vector treatment. Let us now discuss the Picard's method of successive approximations.

25.2 Picard's Method

In this method, a sequence of approximations is constructed by stating with an initial approximation to the solution. The limit of this sequence (if exists) will be the approximation for the solution of the I.V.P given by equations $\frac{dy}{dt} = f(t,y),\ t \in [t_0, b]$

(25.3)

subjected to $(t_0) = y_0$ (25.4)

Integrating (1) between the limits,

$$\int_{y_0}^{y} dy = \int_{t_0}^{t} f(t,y)dt$$

or $y = y_0 + \int_{t_0}^{t} f(t,y)dt$ (25.5)

Notice that the unknown that is to be found isalso seen inside the integral, such an equation is called an integral equation. Such an equation can be solved by the method of successive approximations. To start with the procedure, assume an approximation for the unknown function $y(t)$ as y_0 inside the integral. This makes the equation (25.3) as

$$y = y_0 + \int_{t_0}^{t} f(t, y)dt \tag{25.6}$$

The r.h.s. of (25.4) can be evaluated and call it as $y'(t)$

i.e., $y'(t) = y_0 + \int_{t_0}^{t} f(t, y_0)dt$.

Here $y'(t)$ is the first approximation to the solution. The second approximation to the solution is obtained by using $y'(t)$ in the integral as

$$y^2(t) = y_0 + \int_{t_0}^{t} f(t, y'(t))dt .$$

Repeating this process, we obtain

$$y^n(t) = y_0 + \int_{t_0}^{t} f(t, y^{n-1}(t))dt$$

with $y^{(0)}(t) = y_0$, $n = 1, 2, 3, ...$

By this way we generated a sequence of approximating functions $\left\{ y^n(t) \right\}_{n=1}^{\infty}$.

It can be proved that if the function $f(t, y)$ is bounded in some region about the point (t_0, y_0) and if $f(t, y)$ satisfies the condition $\left| f(t, y) - f(t, y^*) \right| \le k \left| y - y^* \right|$ for the same positive constant k, then the sequence $\left\{ y^n(t) \right\}_{n=1}^{\infty}$ converges to the solution of the I.V.P. given by (25.1)-(25.2) .

Example 1: Find the first three approximate analytical solutions to the I.V.P. $\frac{dy}{dt} = 3t + y^2$ subjected to $y(0) = 1$.

Solution:

Integrating $\frac{dy}{dt} = 3t + y^2$ in the domain, we get

$$\int_1^y dy = \int_0^t (3t + y^2)dt$$

$$\Rightarrow y(t) - y(0) = \int_0^t (3t + y^2)dt$$

or $y(t) = 1 + \int_0^t (3t + y_0{}^2)dt$

or $y'(t) = 1 + \int_0^t (3t + 1)dt$

$$= \frac{3t^2}{2} + t + 1$$

is the first approximation. The second approximation is

$$y^2(t) = 1 + \int_0^t \left[3t + \left(y'(t) \right)^2 \right] dt$$

$$= 1 + \int_0^t \left[3t + \left(\frac{3t^2}{2} + t + 1 \right)^2 \right] dt$$

$$= \frac{9}{20}t^5 + \frac{3}{4}t^4 + \frac{4}{3}t^3 + \frac{5}{2}t^2 + t + 1$$

Likewise, the third approximation can be found as

$$y^3(t) = \frac{81}{4400}x^{11} + \frac{27}{400}x^{10} + \frac{47}{240}x^9 + \frac{17}{32}x^8 + \frac{1157}{1260}x^7 + \frac{68}{45}x^6$$

$$+ \frac{25}{12}x^5 + \frac{23}{12}x^4 + 2x^3 + \frac{5}{2}x^2 + x + 1.$$

The advantage of this method is that we can compute the solution of the given I.V.P. at every point of the domain. The disadvantage of this method is, as also seen in the earlier example, that the integration procedure is laborious and at times, integration might not be possible for some functions $f(t, y)$. The limited utility of this method demands for the search of more elegant methods for solving I.V.Ps.

Example 2: Solve the equation $\frac{dy}{dt} = t + y^2$ subject to the condition $y(t=0) = 1$.

Solution:

Start with $y^{(0)} = y_0 = 1$.

This generates $y^{(1)}(t) = 1 + \int_0^t (t+1)\,dt$

$= 1 + t + \frac{t^2}{2}$

and $y^{(2)}(t) = 1 + \int_0^t \left[t + \left(1 + t + \frac{t^2}{2}\right)^2 \right] dt$

$= 1 + t + \frac{3}{2}t^2 + \frac{2}{3}t^3 + \frac{1}{4}t^4 + \frac{1}{20}t^5$

and so on.

Note that finding $y^{(3)}(t)$ itself involves squaring a 5th degree polynomial and integrating it, thus making this method more tedious.

In the next lesson, we learn one other analytical method that gives an approximation for the solution in a function form.

Exercise:

1. Use Picard's method to find the solution y at $t = 0.2, 0.4$ and 1.0 correct to three decimal places for the I.V.P. $\frac{dy}{dt} = \frac{t^2}{1+y^2}$ subject to $y(0) = 0$ [Hint: obtain $y^{(2)}(t)$ which results in a 9th degree polynomial in t as the approximate solution, which gives the solution correct to 3 decimal places].

2. Use Picard's method successive approximations to solve the following I.V.Ps:

a) $\frac{dy}{dx} = 1 + xy$, $y(0) = 1$.

b) $\frac{dy}{dt} = t - y$, $y(0) = 1$.

c) $\frac{dy}{dt} = t + t^4 y$, $y(0) = 3$.

d) $\frac{dy}{dx} = x + y^2$, $y(0) = 0$.

e) $\frac{dy}{dx} = x + y$, $y(0) = 1$.

Keywords: Approximate analytical solutions, Initial value problems, Picard's Method,

References

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Suggested Reading

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