Lesson 26

Taylor Series Method

26.1 Taylor Series Method

Let $y = y(t)$ be a continuously differentiable function in the interval $[t_0, b]$. Expanding $y(t)$ around $t = t_0$ in Taylor Series, we obtain

$$y(t) = y(t_0) + \frac{(t-t_0)}{1!} y'(t_0) + \frac{(t-t_0)^2}{2!} y''(t_0) + \frac{(t-t_0)^3}{3!} y'''(t_0) + ... \tag{26.1}$$

Taking $t = t_1$ and $t_1 - t_0 = h$, a small increment, i.e., $t_0 + h = t_1, t_1 \in [t_0, b]$ we get

$$y(t_1) = y(t_0) + \frac{h}{1!} y'(t_0) + \frac{h^2}{2!} y''(t_0) + \frac{h^3}{3!} y'''(t_0) + ... \tag{26.2}$$

Now consider the I.V.P. $\frac{dy}{dt} = f(t, y), t \in [t_0, b]$ (26.4)

subject to $y(t_0) = y_0$ (26.5)

The Taylor Series solution of the given I.V.P. (3)-(4) is to find an approximate function $y(t)$ as given in (1) which involves the derivatives of the unknown function $y(t)$ at the initial point $t = t_0$, that satisfies the I.V.P. (3)-(4).

Given $\frac{dy}{dt} = f(t, y)$.

At $t = t_0$; $\frac{dy}{dt}(t = t_0) = f(t_0, y(t_0))$

$= f(t_0, y_0)$

$= f_0$ (say)

$$y''(t) = \frac{d}{dt}\left(\frac{dy}{dt}\right)$$

$$= \frac{d}{dt}\left(f(t,y)\right) = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

$$= \frac{\partial f}{\partial t} + \frac{\partial f}{\partial y}\left(f(t,y)\right).$$

$$y''(t=t_0) = \frac{\partial f}{\partial t}\left(t=t_0, y(t_0)\right) + \frac{\partial f}{\partial y}\left(t=t_0, y(t_0)\right)\cdot f\left(t=t_0, y(t_0)\right).$$

In the same manner, we calculate the higher order derivatives of $y(t)$ depending on the necessity. Before we discuss the order of approximation of the Taylor Series method and error associated with a particular order method, let us see its utility for finding the solution of the given I.V.P. through a few examples.

Example 1: Find $y(0.1)$ from the I.V.P. $\frac{dy}{dt} = 3t + y^2$ subject to $y(0) = 1$.

Solution:

Given $y' = 3t + y^2$

$$\Rightarrow y'(0) = 0 + \left[y(0)\right]^2 = 1$$

$$y'' = 3 + 2y\cdot y' \Rightarrow y''(0) = 3 + 2y(0)\cdot y'(0)$$

$$= 3 + 2\cdot 1\cdot 1 = 5$$

$$y''' = 2y\cdot y'' + 2y'^2 \Rightarrow y'''(0) = 2y(0)\cdot y''(0) + 2\left[y'(0)\right]^2$$

$$= 2\cdot 1\cdot 5 + 2\cdot 1 = 12$$

$$y^{iv} = 2y'\cdot y'' + 2y\cdot y''' + 4y'\cdot y'' = 2\cdot 1\cdot 5 + 2\cdot 1\cdot 12 + 4\cdot 1\cdot 5 = 54 \text{ etc.}$$

Now the Taylor Series solution is written as

$$y(t)=y(t_0)+(t-t_0)\,y'(t_0)+\frac{(t-t_0)^2}{2!}y''(t_0)+\frac{(t-t_0)^3}{3!}y'''(t_0)+\frac{(t-t_0)^4}{4!}y^{iv}(t_0)+\ldots$$

Taking $t=0.1, t_0=0$ (given),

$$y(0.1)=y(0)+(0.1)y'(0)+\frac{(0.1)^2}{2!}y''(0)+\frac{(0.1)^3}{3!}y'''(0)+\frac{(0.1)^4}{4!}y^{iv}(0)+\ldots$$

$$=1+0.1+\frac{5}{2}(0.1)^2+\frac{12}{3!}(0.1)^3+\frac{54}{4!}(0.1)^4+\ldots$$

$\therefore\ y(0)=1.12722$.

Example 2: Obtain the Taylor Series solution for the I.V.P. given by $\frac{dy}{dt}=2y+3e^t$, $y(0)=0$. Compare the solution at $t=0.2$ with the exact solution given by $y(t)=3\left(e^{2t}-e^t\right)$.

Solution:

Given $y'=2y+3e^t \Rightarrow y'(0)=3$

$y''=2y'+3e^t \Rightarrow y''(0)=9$

$y'''=2y''+3e^t \Rightarrow y'''(0)=21$

$y^{iv}=2y'''+3e^t \Rightarrow y^{iv}(0)=45$

The solution is $y(t)=3t+\frac{9}{2}t^2+\frac{21}{6}t^3+\frac{45}{24}t^4+\ldots$.

At $t=0.2$, $y(0.2)=0.811$.

From the exact solution, $y(0.2)=0.8112$.

So the error in the numerical solution is $|0.8112-0.811|=0.001$.

26.2 Local Truncation Error and Order of the Method

The Taylor series expansion of $y(t)$ about any point t_j is written as

$$y(t) = y(t_9) + (t - t_9)\, y'(t_9) + \frac{(t - t_9)^2}{2!} y''(t_9) + \ldots$$

$$\ldots + \frac{1}{p!}(t - t_0)^p\, y^{(p)}(t_9) + \frac{1}{(p+1)!}(t - t_9)^{(p+1)}\, y^{(p+1)}(t_9 + \theta h) \qquad (26.6)$$

where $y^{(p)}(t)$ is the pth of the $y(t)$ and $0 < \theta < 1$, a real number. The last term in the expansion is called the remainder term. In general, the local truncation error at any location $t = t_{j+1}$ of the method is given by

$$T_{j+1} = \frac{1}{(p+1)!} h^{(p+1)} y^{(p+1)}(t_j + \theta h)$$

where $h = t_{j+1} - t_j$.

The order of a method is the largest integer p for which $\left|\frac{1}{h} T_{j+1}\right| = O(h^p)$.

The notation $O(h^p)$ denoting that all terms of the order p onwards are grouped to a single term representation.

The method given by equation (26.1) is called the Taylor Series method of order p.

Example 3: Determine the first three non-zero terms in the Taylor Series for $y(t)$ from the I.V.P. $y' = t^2 + y^2$, $y(0) = 0$.

Solution:

$y'(0) = 0;$

$y'' = 2t + 2yy' \Rightarrow y''(0) = 0$

$y''' = 2 + 2\left(y'\right)^2 + 2yy'' \Rightarrow y'''(0) = 2$

$y^{iv} = 6y'y'' + 2yy''' \Rightarrow y^{iv}(0) = 0$

$y^{v} = 2yy^{v} + 8y'y''' + 6\left(y''\right)^2 \Rightarrow y^{v}(0) = 0$

$y^{vi} = 2yy^{v} + 10y'y^{iv} + 20y''y''' \Rightarrow y^{vi}(0) = 0$

Similarly $y^{vii}(0) = 80$, $y^{viii}(0) = 0 = y^{ix}(0) = y^{x}(0)$, $y^{xi}(0) = 38400$.

Thus the three term Taylor Series solution for the given I.V.P. is

$y(t) = \frac{1}{3}t^3 + \frac{1}{63}t^7 + \frac{2}{2079}t^{11}$.

Example 4: Given the I.V.P.: $y' = 2t + 3y$; $y(0) = 1$ whose exact solution is

$y(t) = \frac{11}{9}e^{3t} - \frac{2}{9}\left(3t + 1\right)$.

a) Use 2nd order Taylor Series method to get $y(0.2)$ with step length $h = 0.1$ (note $h = t_{j+1} - t_j$).

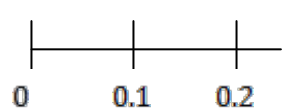
b) Find 't', if the error in $y(t)$ obtained from the first four terms of the Taylor series, is to be less than 5×10^{-5}, after rounding.

c) Determine the number of terms in the Taylor Series required to obtain the result correct to 5×10^{-6} for $t \le 0.4$.

Solution:

a) The second order Taylor Series method is given by $y_{n+1} = y_n + hy'_n + \frac{h^2}{2}y''_n + O\left(h^3\right)$.

$n = 0,1; h = 0.1$.



We have $y' = 2t + 3y$

or, $y'(t_n) = y'_n = 2t_n + 3y_n$

or, $y''(t_n) = y''_n = 2 + 3y'_n = 2 + 6t_n + 9y_n$.

Take $n = 0; h = 0.1; y_0 = 1; y'_0 = 3; y''_0 = 11$.

$\therefore y(0.1) = 1 + (0.1)\cdot 3 + \frac{(0.1)^2}{2}\cdot 11 = 1.355$.

Taking $n = 1; h = 0.1; y_1 = 1.355; y'_1 = 4.265; y''_1 = 14.795$

$\Rightarrow y(0.2) = 1.355 + (0.1)\cdot(4.265) + \frac{(0.1)^2}{2!}\cdot(14.795)$

$= 1.8555$.

b) Given $y(0) = 1, y'(0) = 3$.

Compute $y''(0) = 11, y'''(0) = 33$, $y^{iv}(0) = 99$.

So the Four term Taylor Series solution is

$y(t) = 1 + 3t + \frac{11}{2}t^2 + \frac{11}{2}t^3$.

The remainder term is given by

$R_4 = \frac{t^4}{4!}y^{iv}(\xi)$ gives the error in the approximation.

We require $|R_4| \le 5\times 10^{-5}$.

Given the exact solution, use it to find $y^{iv}(\xi)$;

$y(t) = \frac{11}{9}e^{3t} - \frac{2}{9}(3t+1)$.

$y^{iv}(\xi) = 99e^{3t}$.

$\therefore |R_4| = \left|\frac{99}{24}e^{3t}t^4\right| < 5\times 10^{-5}$.

Simplifying and solving this non-linear algebraic equation for finding t, we get

$t^4 e^{3t} < 0.000012$

$\Rightarrow t < 0.056$

Note: In the event, if the exact solution is not known, write one more non-vanishing term in the Taylor Series than is required and then differentiate this series p times: here p is 4.

Now to determine the number of terms required in the Taylor series solution to obtain the solution correct to 5×10^{-4} for $t \le 0.4$, we estimate it as:

$$\max_{0\le t\le 0.4}\left|\frac{t^p}{p!}\right|\cdot \max_{\xi t[0,0.4]}\left|y^{(p)}(\xi)\right| \le 5\times10^{-6}$$

Again using the analytical solution, we find the pth derivative of $y(t)$ at $t=\xi$ as:

$$y^p(\xi) = (11)3^{p-2}\cdot e^{1.2}$$

or $\dfrac{(0.4)^p}{p!}(11)3^{p-2}\cdot e^{1.2} \le 5\times10^{-6}$

Solving this non-linear algebraic equation using the Newton-Raphson method or otherwise, we get a lower bound for p as

$$p \ge 10$$

This indicates that a minimum of 10th order Taylor Series method gives the solution which will be accurate upto the 6th decimal place for all values of $t \in [0, 0.4]$.

Exercises:

1. Compute an approximation to $y(0.1)$ up to five decimal places from

   $\dfrac{dy}{dt} = t^2 y - 1, y(0) = 1$.

2. Solve $y' = y^2 + t$, $y(0) = 1$ using Taylor Series method and compute $y(0.2)$, $h = 0.1$.

3. Evaluate $y(0.1)$ correct to six decimal places $\left(5\times10^{-6}\right)$ by Taylor Series method if $y(t)$ satisfies $\frac{dy}{dt}=1+yt, y(0)=1$.

Keywords: Taylor Series method, Taylor series solution.

References

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Suggested Reading

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