Lesson 28

Modified Euler Method

28.1 Introduction

The first order Explicit Euler method is improved to achieve better accuracy in the numerical solution for an initial value problem $y' = f(t, y), y(t_0) = y_0$.

In the modified Euler method, the slope of $y(t)$ at $t = t_j$ is approximated by the average of the slopes at $t = t_j$ and $t = t_{j+1}$.

i.e., $y_{j+1} = y_j + h \cdot y'_j$

$$= y_j + \frac{h}{2} \cdot \left[f\left(t_j, y_j\right) + f\left(t_{j+1}, y_{j+1}\right) \right]$$

or $$y_{j+1} = y_j + \frac{h}{2} \cdot \left[f\left(t_j, y_j\right) + f\left(t_{j+1}, y_{j+1}\right) \right] \tag{28.1}$$

This implicit method is used by setting an iterative procedure as follows:

$$y_{j+1}^{(s+1)} = y_j + \frac{h}{2} \cdot \left[f\left(t_j, y_j\right) + f\left(t_{j+1}, y_{j+1}^{(s)}\right) \right], \; s = 0, 1, 2, \ldots \tag{28.2}$$

Now the initial approximation for $y_{j+1}^{(0)}$ is considered as the solution y_{j+1} of Euler method. The above iteration process is terminated at each step if the condition

$$\left| y_j^{(s+1)} - y_j^{(s)} \right| < \varepsilon \text{,is satisfied,}$$

where ε is a reassigned error tolerance.

Exercise: Show that the modified Euler method has L.T.E. as $O(h^3)$ and the order of the method is $O(h^2)$.

The exponent of h in $O(h^p)$ is the order of accuracy of the method. It is a measure of accuracy of any numerical scheme. It gives an indication of how

rapidly the accuracy can be improved with refinement of the grid spacing h in any given interval. For example, in a first order method such as $y_{j+1} = y_j + h \cdot f\left(t_j, y_j\right) + O(h)$

If we reduce the mesh size h by $\frac{h}{2}$, the error is reduced by approximately a factor $\frac{1}{2}$. Similarly in a second order method such as

$$y_{j+1} = y_j + \frac{h}{2} \cdot \left[f\left(t_j, y_j\right) + f\left(t_{j+1}, y_{j+1}\right) \right] + O(h^2).$$

If we refine the mesh size by a factor of 2, we expect the error to reduce by a factor 2^2 i.e., 4, which gives a rapid decrease in the error. Thus higher order numerical schemes are preferred.

Example 1: Determine the value of $y(0.1)$ from the I.V.P. $y' = y + t^2$, $y(0) = 1, h = 0.05$.

Solution:

$t_0 = 0, t_1 = t_0 + h = 0.05, t_2 = t_0 + 2h = t_1 + h = 0.1$, $y_0 = 0$, $f(t, y) = t^2 + y$; $y_1 = y(0.05)$, $y_2 = y(0.1)$.

$$y_1^{(s+1)} = y_0 + \frac{h}{2} \cdot \left[f\left(t_0, y_0\right) + f\left(t_1, y_1^{(s)}\right) \right], \; s = 0, 1, 2, \ldots$$

Compute $y_1^{(0)}$ using the Euler method.

$$y_1^{(0)} = y_0 + \frac{h}{2} f\left(t_0, y_0\right)$$

$$= 1 + (0.05)(1.0)$$

$$= 1.05.$$

Use modified Euler method now as follows:

Compute $y(0.05)$:

Take $s=0$: calculate $f\left(t_1, y_1^{(0)}\right)=1.0262$

$$y_1^{(1)}=y_0+\frac{h}{2}\cdot\left[f\left(t_0, y_0\right)+f\left(t_1, y_1^{(0)}\right)\right]$$

$=1.0513$

Take $s=1$; $y_1^{(2)}$ is computed as:

$$y_1^{(2)}=y_0+\frac{h}{2}\cdot\left[f\left(t_0, y_0\right)+f\left(t_1, y_1^{(1)}\right)\right]$$

$=1.0513$

Note: $\left|y_1^{(2)}-y_1^{(1)}\right|=0$, so we can stop the iteration process and conclude that $y(0.05)=1.0513$.

Compute $y(1.0)$: $t_1=0.05, y_1=1.0513, h=0.05$

Euler method gives $y_2^{(0)}=y_1+hf\left(t_1, y_1\right)$

$\Rightarrow y_2^{(0)}=1.104$.

Now use the modified Euler method:

$$s=0 \Rightarrow y_2^{(1)}=y_1+\frac{h}{2}\cdot\left[f\left(t_1, y_1\right)+f\left(t_2, y_2^{(0)}\right)\right]$$

$=1.1055$

$$s=1 \Rightarrow y_2^{(2)}=y_1+\frac{h}{2}\cdot\left[f\left(t_1, y_1\right)+f\left(t_2, y_2^{(1)}\right)\right]$$

$=1.1055$

Take the solution $y(1.0)=1.1055$.

Example 2: Given the I.V.P. $\frac{dy}{dt}=2ty, y(1)=1$ find $y(1.4)$ using the modified Euler method by taking $h=0.1$. Compare this solution with the exact solution $y(t)=e^{t^2-1}$. Calculate the percentage relative error.

Solution:

We have the data: $t_0 = 1, t_1 = 1.1, t_2 = 1.2, t_3 = 1.3, t_4 = 1.4, y_0 = 1, h = 0.1$.

Modified Euler method is:

$$y_{j+1} = y_j + \frac{h}{2} \cdot \left[f\left(t_j, y_j\right) + f\left(t_{j+1}, y_{j+1}\right) \right] \text{for } j = 0,1,2,3.$$

Take $j = 0$: We have the iterative method is written as

$$y_1^{(s+1)} = y_0 + \frac{h}{2} \cdot \left[f\left(t_0, y_0\right) + f\left(t_1, y_1^{(s)}\right) \right], \; s = 0,1,2,\ldots$$

Compute $y_1^{(0)}$ using the Euler method:

$$y_1^{(0)} = y_0 + (0.1) \cdot 2 \cdot (0.1) \cdot 1 = 1.2.$$

Now $y_1^{(1)} = y_0 + \frac{(0.1)}{2}\left[2t_0 y_0 + 2t_1 y_1^{(0)}\right] = 1.232$,

$$y_1^{(2)} = y_0 + \frac{(0.1)}{2}\left[2t_0 y_0 + 2t_1 y_1^{(1)}\right] = 1.232.$$

$j = 1$ to $j = 3$ are calculated (left as an exercise) and are tabulated below. The absolute error and percentage relative errors are calculated as:

$Absolute\ error = |Exact\ value - Numerical\ Value|$ and

$ntage\ relative\ error = \frac{|error|}{Exact\ value}$.

Table 28.1

j	t_n	y_n	Exact value $y = e^{t_n^2 - 1}$	Absolute error	Percentage relative error
0	1	1	1	0	0
1	1.1	1.232	1.2337	0.0017	0.14
2	1.2	1.5479	1.5527	0.0048	0.31
3	1.3	1.9832	1.9937	0.0106	0.53
4	1.4	1.5908	2.6117	0.0209	0.80

Exercises:

1. Solve the I.V.Ps. using modified Euler method

(i) $\frac{dy}{dt} = -2y, y(0) = 1, h = 0.2$, $t \in [0, 0.6]$.

(ii) $\frac{dy}{dt} = \frac{y-t}{y+t}, y(0) = 1, h = 0.1, t \in [0, 0.2]$.

(iii) $\frac{dy}{dt} = 2 + \sqrt{|ty|}, y(1) = 1, h = 0.5, t \in [1, 2]$.

(iv) $y' = t(1 + t^3 y), y(0) = 3, h = 0.1, t \in [0, 0.4]$.

Keywords: Absolute error, Explicit Euler method, Percentage relative errors, Modified Euler method,

References

Jain. M. K., Iyengar. S.R.K., Jain. R.K.,(2008).Numerical Methods. Fifth Edition, New Age International Publishers, New Delhi.

Atkinson. E Kendall, (2004). Numerical Analysis. Second Edition, John Wiley & Sons, Publishers, Singapore.

Suggested Reading

Scheid.Francis,(1989). Numerical Analyysis. Second Edition, Mc Graw-Hill Publishers, New York.

Sastry.S.S, (2005). Introductory Methods of Numerical Analysis. Fourth Edition,Prentice Hall of India Publishers, New Delhi.