Lesson 29

Runge-Kutta Methods

29.1 Introduction

In single step explicit method, the approximate solution y_{j+1} is computed from the known solution at the point (t_j, y_j) using

$$y_{j+1} = y_j + h \cdot f(t_j, y_j) \tag{29.1}$$

or $y_{j+1} = y_j + h$ (slope of $y(t)$ at $t = t_j$) (29.2)

In equation (1), we used the slope at $t = t_j$ only. Similarly, in the modified Euler method

$$y_{j+1} = y_j + h\left[f(t_j, y_j) + f(t_{j+1}, y_{j+1}) \right] \tag{29.3}$$

The slope is replaced by the average of slopes at the end points (t_j, y_j) and (t_{j+1}, y_{j+1}).

29.2 Runge-Kutta Methods

Runge-Kutta methods use a weighted average of slopes on the given interval $[t_j, t_{j+1}]$, instead of a single slope. Thus the general Runge-Kutta method may be defined as

$y_{j+1} = y_j + h$ [Weighted average of slopes at n points on the given interval] (29.4)

This way one can derive N-explicit methods by taking $n = 1, 2, ..., N$. Also, n in (29.4) indicates the order of this Runge-Kutta method. The general N^{th} order

Runge-Kutta method is written as

$$y_{j+1} = y_j + \left(w_1 k_1 + w_2 k_2 + \ldots + w_N k_N\right)$$

where w_i are the weights and each k_i is defined as

$$k_1 = h \cdot f\left(t_j, y_j\right)$$

$$k_2 = h \cdot f\left(t_j + c_2 h, y_j + a_{21} k_1\right)$$

$$k_3 = h \cdot f\left(t_j + c_3 h, y_j + a_{31} k_1 + a_{32} k_2\right)$$

$$k_N = h \cdot f\left(t_j + c_N h, y_j + a_{N1} k_1 + a_{N2} k_2 + \ldots + a_{N,N-1} k_{N-1}\right) \tag{29.5}$$

All w_i's , c_i's, a_i's are the parameters which are determined by forcing the method (29.4) to be of N^{th}order. Deriving a general N^{th} order method is out of purview of this material, but we demonstrate the derivation of the 2nd order Runge-Kutta method below.

29.3 Second Order Runge-Kutta Method

Consider the general form of the (2nd order) Runge-Kutta method with 2 slopes,

$$y_{j+1} = y_j + w_1 k_1 + w_2 k_2 \tag{29.6}$$

where $k_1 = h \cdot f\left(t_j, y_j\right)$

$$k_2 = h \cdot f\left(t_j + c_2 h, y_j + a_{21} k_1\right)$$

The parameters c_2, a_{21}, w_1, w_2 are chosen to make y_{j+1} closer to the exact solution $y(t_{j+1})$ upto the 2nd order.

Now writing $y(t_{j+1})$ in Taylor series about $t = t_j$,

$$y(t_{j+1}) = y(t_j) + h \cdot y'(t_j) + \frac{h^2}{2!} \cdot y''(t_j) + \ldots$$

$$= y(t_j) + h \cdot f\left(t_j, y_j\right) + \frac{h^2}{2!} \cdot \left(\frac{\partial f}{\partial t} + f \frac{\partial f}{\partial y} \right)\Bigg|_{t=t_j} + \ldots \qquad (29.7)$$

Also, $k_1 = h \cdot f_j$

$$k_2 = h \cdot f\left(t_j + c_2 h, y_j + a_{21} h \cdot f_j\right)$$

Expanding f about $\left(t_j, y_j\right)$, we get

$$k_2 = h\left[f\left(t_j\right) + h\left(c_2 \frac{\partial f}{\partial t} + a_{21} f \frac{\partial f}{\partial y} \right)\Bigg|_{t=t_j} + \frac{h^2}{2!} \cdot \left(c_2^2 \frac{\partial^2 f}{\partial t^2} + 2c_2 a_{21} f \frac{\partial^2 f}{\partial y \partial t} + a_{21} f \frac{\partial^2 f}{\partial y^2} \right)\Bigg|_{t=t_j} + \ldots \right]$$

Substituting the expressions for k_1 and k_2 in (1),we get

$$y_{j+1} = y_j + \left(w_1 + w_2\right) h \cdot f_j + h^2 \left(w_2 c_2 \frac{\partial f}{\partial t} + w_2 a_{21} f \frac{\partial f}{\partial y} \right)\Bigg|_{t=t_j} + \ldots \qquad (29.8)$$

Comparing the coefficients of h and h^2 in (27.7) and (27.8) we obtain

$$w_1 + w_2 = 1; \; w_2 c_2 = \frac{1}{2}; w_2 a_{21} = \frac{1}{2}.$$

The solution of this may be written as

$$a_{21} = c_2; w_2 = \frac{1}{2c_2}; w_1 = 1 - \frac{1}{2c_2},$$

c_2 is arbitrary non-zero constant.

With the choice of $c_2 = \frac{1}{2}$, we derive the 2nd order Runge-Kutta method.

$a_{21} = \frac{1}{2}; w_2 = 1; w_1 = 0$

we get $k_1 = h \cdot f\left(t_j, y_j\right)$

$k_2 = h \cdot f\left(t_j + \frac{h}{2}, y_j + \frac{1}{2}k_1\right)$

And $y_{j+1} = y_j + k_2$ (29.9)

With the choice of $c_2 = 1$, we get

$w_2 = \frac{1}{2}, w_1 = \frac{1}{2}$ and $a_{21} = 1$

and $y_{j+1} = y_j + \frac{1}{2}(k_1 + k_2)$

where $k_1 = h \cdot f\left(t_j, y_j\right)$

$k_2 = h \cdot f\left(t_j + h, y_j + k_1\right)$.

This method is also a second order method which is known as the Euler-Cauchy method. Clearly, with different choices for c_2 , we get a different second order Runge-Kutta method. Let us demonstrate its utility for solving the initial value problems.

Example1: Compute $y(0.4)$ from the I.V.P. $y' = -2ty^2, y(0) = 1, h = 0.2$

Solution:

$y_{j+1} = y_j + k_2$

where $k_1 = h \cdot f\left(t_j, y_j\right) = (0.2)\left[-2t_j y_j^{\,2}\right] = -0.4t_j y_j^{\,2}$

$k_2 = h \cdot f\left(t_j + \frac{h}{2}, y_j + \frac{1}{2}k_1\right)$

$= -0.4\left(t_j + 0.1\right)\left(y_j + \frac{1}{2}k_1\right)^2$

Taking $j = 0$; given that $t_0 = 0, y_0 = 1$,

$\Rightarrow k_1 = 0, k_2 = -0.04$.

$\therefore y(0.2) = y(0) + k_2 = 1 - 0.04 = 0.96$.

For $j = 1$, $t_1 = 0.2, y_1 = 0.96$

$\Rightarrow k_1 = -0.073728, k_2 = -0.10226$

and $y(0.4) = y(0.2) + k_2 = 0.96 - 0.10226 = 0.85774$

Keywords: Weighted average of slopes,

References

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Suggested Reading

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