Lesson 38

Properties of Laplace Transform (Cont.)

In this lesson we further continue discussing properties of Laplace transform. In particular, this lesson is devoted to Laplace transform of functions which are multiplied and divide by t.

38.1 Multiplication by t^n

38.1.1 Theorem

If $F(s)$ is the Laplace transform of $f(t)$, i.e., $L[f(t)] = F(s)$ then,

$$L[tf(t)] = -\frac{d}{ds}F(s)$$

and in general the following result holds

$$L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}F(s).$$

Proof: By definition we know

$$F(s) = \int_0^\infty e^{-st} f(t)\,\mathrm{d}t$$

Using Leibnitz rule for differentiation under integral sign we obtain

$$\frac{dF(s)}{ds} = \int_0^\infty (-t)e^{-st} f(t)\,\mathrm{d}t$$

Thus we get

$$\frac{dF(s)}{ds} = -L[tf(t)]$$

Repeated differentiation under integral sign gives the general rule. ■

Applicability of the above result will now be demonstrated by some examples.

38.2 Example Problems

38.2.1 Problem 1

Find Laplace transform of the function $t^2 \cos at$.

Solution: We know from Laplace transform of elementary functions that

$$L[\cos at] = \frac{s}{s^2 + a^2}$$

Direct application of the above rule gives

$$L\left[t^2 \cos at\right] = \frac{d^2}{ds^2}\left(\frac{s}{s^2 + a^2}\right) = \frac{d}{ds}\left(\frac{s^2 + a^2 - 2s^2}{\left(s^2 + a^2\right)^2}\right) = \frac{d}{ds}\left(\frac{a^2 - s^2}{\left(s^2 + a^2\right)^2}\right)$$

On simplifications we find

$$L\left[t^2 \cos at\right] = \frac{2s\left(s^2 - 3a^2\right)}{\left(s^2 + a^2\right)^3}$$

38.2.2 Problem 2

Evaluate (i) $L[te^{-t}]$ *(ii)* $L[t^2 e^{-t}]$ *(iii)* $L[t^k e^{-t}]$

Solution: (i) We know that

$$L[e^{-t}] = \frac{1}{s+1}$$

Using the above mentioned rule we find

$$L[te^{-t}] = -\frac{d}{ds}\frac{1}{s+1} = \frac{1}{(s+1)^2}$$

(ii) Applying the same idea once again, we obtain

$$L[t^2 e^{-t}] = -\frac{d}{ds}\frac{1}{(s+1)^2} = \frac{2}{(s+1)^3}$$

(iii) Similarly, we can further generalize this result as

$$L[t^k e^{-t}] = \frac{k!}{(s+1)^{k+1}}$$

38.2.3 Problem 3

Find the Laplace transform of $f(t) = (t^2 - 3t + 2)\sin t$

Solution: Using linearity of the Laplace transform we have

$$L[f(t)] = L[t^2 \sin t] - 3L[t \sin t] + 2L[\sin t] \tag{38.1}$$

Since we know

$$L[\sin t] = \frac{1}{1+s^2}$$

then

$$L[t \sin t] = -\frac{d}{ds}\frac{1}{1+s^2} = \frac{2s}{(1+s^2)^2}$$

and

$$L[t^2 \sin t] = -\frac{d}{ds}\frac{2s}{(1+s^2)^2} = \frac{2(1+s^2)^2 - 8s^2(1+s^2)}{(1+s^2)^4} = \frac{6s^2-2}{(1+s^2)^3}$$

Substituting the above values in the equation (38.1), we find

$$L[f(t)] = \frac{6s^2-2}{(1+s^2)^3} - \frac{6s}{(1+s^2)^2} + \frac{2}{1+s^2}$$

Further simplifications lead to

$$L[f(t)] = \frac{6s^2 - 2 - 6s(1+s^2) + 2(1+s^2)^2}{(1+s^2)^3}$$

Finally, we obtain

$$L[f(t)] = \frac{(2s^4 - 6s^3 + 10s^2 - 6s)}{(s^6 + 3s^4 + 3s^2 + 1)}$$

38.3 Division by t

38.3.1 Theorem

If f *is piecewise continuous on* $[0, \infty)$ *and is of exponential order* α *such that*

$$\lim_{t \to 0+} \frac{f(t)}{t}$$

exists, then,

$$L\left[\frac{f(t)}{t}\right] = \int_s^\infty F(u)\,\mathrm{d}u, \quad [s > \alpha]$$

Proof: This can easily be proved by letting $g(t) = \dfrac{f(t)}{t}$ so that $f(t) = tg(t)$.

Hence,

$$F(s) = L[f(t) = L[tg(t)] = \frac{d}{ds}L[g(t)]$$

Integrating with respect to s we get,

$$-L[g(t)]\Big|_s^\infty = \int_s^\infty F(s)\,\mathrm{d}s.$$

Since $g(t)$ is piecewise continuous and of exponential order, it follows that $\lim\limits_{s\to\infty} L[g(t)] \to 0$. Thus we have

$$L[g(t)] = \int_0^\infty F(s)\,\mathrm{d}s.$$

This completes the proof. ■

Remark: *It should be noted that the condition* $\lim\limits_{t\to 0+}[f(t)/t]$ *is very important because without this condition the function* $g(t)$ *may not be piecewise continuous on* $[0,\infty)$*. Thus without this condition we can not use the fact* $\lim\limits_{s\to\infty} L[g(t)] \to 0$.

38.3.2 Corollary

If $L[f(t)] = F(s)$ *then* $\displaystyle\int_0^\infty \frac{f(t)}{t}\,dt = \int_0^\infty f(s)\,ds$*, provided that the integrals converge.*

Proof: We know that

$$L\left[\frac{f(t)}{t}\right] = \int_s^\infty F(u)\,\mathrm{d}u$$

Using the definition of Laplace transform we get

$$\int_0^\infty e^{-st}\frac{f(t)}{t}\,\mathrm{d}t = \int_s^\infty F(u)\,\mathrm{d}u$$

Taking limit $s \to 0$ in above two integrals we obtain

$$\int_0^\infty \frac{f(t)}{t}\,\mathrm{d}t = \int_0^\infty F(u)\,\mathrm{d}u$$

This completes the proof. ■

38.4 Example Problems

38.4.1 Problem 1

Find the Laplace transform of the function

$$f(t) = \frac{\sin at}{t}$$

Solution: We know,

$$L[\sin at] = \frac{a}{s^2 + a^2} \quad \text{and} \quad L\left[\frac{f(t)}{t}\right] = \int_s^\infty F(u)\,\mathrm{d}u$$

On integrating we get,

$$L\left[\frac{\sin at}{t}\right] = \int_s^\infty \frac{a}{s^2 + a^2}\,\mathrm{d}s = \tan^{-1}\left(\frac{s}{a}\right)\Big|_s^\infty$$

Thus we have

$$L\left[\frac{\sin at}{t}\right] = \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{a}\right)$$

38.4.2 Problem 2

Find the Laplace transform of the function

$$f(t) = \frac{2\sin t \sinh t}{t}$$

Solution: Using Division by t property of the Laplace transform we get

$$L[f(t)] = \int_s^\infty L\left[\sin t\left(e^t - e^{-t}\right)\right]\,\mathrm{d}s \tag{38.2}$$

Now we evaluate $L\left[\sin t\left(e^t - e^{-t}\right)\right]$ using linearity of the Laplace transform as

$$L\left[\sin t\left(e^t - e^{-t}\right)\right] = L\left[e^t \sin t\right] - L\left[e^{-t}\sin t\right]$$

Applying the first shifting theorem we obtain

$$L\left[\sin t\left(e^t - e^{-t}\right)\right] = \frac{1}{1 + (s-1)^2} - \frac{1}{1 + (s+1)^2}$$

Substituting this value in the equation (38.2) we find

$$L[f(t)] = \int_s^\infty \left[\frac{1}{1+(s-1)^2} - \frac{1}{1+(s+1)^2}\right] \mathrm{d}s$$

On integrating, we have

$$\begin{aligned} L[f(t)] =& \tan^{-1}(s-1)\Big|_s^\infty - \tan^{-1}(s+1)\Big|_s^\infty \\ =& \frac{\pi}{2} - \tan^{-1}(s-1) - \frac{\pi}{2} + \tan^{-1}(s+1) \end{aligned}$$

On cancellation of $\pi/2$ we get

$$L[f(t)] = \tan^{-1}(s+1) - \tan^{-1}(s-1)$$

This can be further simplified to obtain

$$L[f(t)] = \tan^{-1}\left(\frac{2}{s^2}\right)$$

Suggested Readings

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