Lesson 43

Laplace Transform of Some Special Functions

In this lesson we discuss Laplace transform of some special functions like error functions, Dirac delta functions, etc. There functions appears in various applications of science and engineering to some of them we shall encounter while solving differential equations using Laplace transform.

43.1 Error Function

The error appears in probability, statistics and solutions of some partial differential equations. It is defined as

$$\text{erf}(t) = \frac{2}{\sqrt{\pi}} \int_0^t e^{-u^2}\, \mathrm{d}u$$

Its complement, known as complementary error function, is defined as

$$\text{erfc}(t) = 1 - \text{erf}(t) = \frac{2}{\sqrt{\pi}} \int_t^\infty e^{-u^2}\, \mathrm{d}u$$

We find Laplace transform of different forms of error function in the following examples.

43.2 Example Problems

43.2.1 Problem 1

Find $L[\textit{erf}(\sqrt{t})]$.

Solution: From definition of the error function and the Laplace transform we have,

$$L[\text{erf}(\sqrt{t})] = \frac{2}{\sqrt{\pi}} \int_0^\infty \int_0^{\sqrt{t}} e^{-st} e^{-x^2} \mathrm{d}x \mathrm{d}t$$

By changing the order of integration we get,

$$L[\text{erf}(\sqrt{t})] = \frac{2}{\sqrt{\pi}} \int_{x=0}^\infty \int_{t=x^2}^\infty e^{-st} e^{-x^2} \mathrm{d}t \mathrm{d}x$$

Evaluating the inner integral we obtain

$$L[\text{erf}(\sqrt{t})] = \frac{2}{\sqrt{\pi}} \int_{x=0}^{\infty} e^{-x^2} \frac{e^{-sx^2}}{s} \mathrm{d}x = \frac{2}{\sqrt{\pi}} \frac{1}{s} \int_{x=0}^{\infty} e^{-(1+s)x^2} \mathrm{d}x$$

Substituting $\sqrt{(1+s)}x = u \Rightarrow dx = \dfrac{1}{\sqrt{1+s}} du$

$$L[\text{erf}(\sqrt{t})] = \frac{2}{\sqrt{\pi}} \frac{1}{s\sqrt{1+s}} \int_{x=0}^{\infty} e^{-u^2} \mathrm{d}u = \frac{1}{s\sqrt{s+1}}$$

Note that we have used the value of Gaussian integral $\displaystyle\int_{x=0}^{\infty} e^{-u^2} \mathrm{d}u = \frac{\sqrt{\pi}}{2}$.

43.2.2 Problem 2

Find $L\left[\textit{erf}\left(\dfrac{k}{\sqrt{t}}\right)\right]$. *and show that* $L^{-1}\left[\dfrac{e^{-2k\sqrt{s}}}{s}\right] = \textit{erfc}\left(\dfrac{k}{\sqrt{t}}\right)$

Solution: By the definition of Laplace transform we have

$$L\left[\text{erf}\left(\frac{k}{\sqrt{t}}\right)\right] = \int_0^{\infty} e^{-st} \frac{2}{\sqrt{\pi}} \int_0^{\frac{k}{\sqrt{t}}} e^{-u^2} \mathrm{d}u \, \mathrm{d}t$$

Changing the order of integration we get

$$L\left[\text{erf}\left(\frac{k}{\sqrt{t}}\right)\right] = \frac{2}{\sqrt{\pi}} \int_0^{\infty} \int_0^{\frac{k^2}{u^2}} e^{-st} e^{-u^2} \mathrm{d}t \, \mathrm{d}u$$

Evaluation of the inner integral leads to

$$L\left[\text{erf}\left(\frac{k}{\sqrt{t}}\right)\right] = \frac{2}{\sqrt{\pi}} \frac{1}{s} \int_0^{\infty} e^{-u^2} \left(1 - e^{-s\frac{k^2}{u^2}}\right) \mathrm{d}u$$

Using the value of Gaussian integral we have

$$L\left[\text{erf}\left(\frac{k}{\sqrt{t}}\right)\right] = \frac{2}{\sqrt{\pi}} \frac{1}{s} \left[\frac{\sqrt{\pi}}{2} - \int_0^{\infty} \left(e^{-u^2 - s\frac{k^2}{u^2}}\right) \mathrm{d}u\right] \tag{43.1}$$

Let us assume

$$I(s) = \int_0^{\infty} e^{-u^2 - s\frac{k^2}{u^2}} \mathrm{d}u$$

By differentiation under integral sign

$$\frac{dI}{ds} = \int_0^{\infty} e^{-u^2 - s\frac{k^2}{u^2}} \left(-\frac{k^2}{u^2}\right) \mathrm{d}u$$

Substitution $\frac{\sqrt{s}k}{u} = x \Rightarrow -\frac{\sqrt{s}k}{u^2}du = dx$ leads to

$$\frac{dI}{ds} = -\frac{k}{\sqrt{s}}\int_0^\infty e^{-x^2 - s\frac{k^2}{x^2}}\mathrm{d}x = -\frac{k}{\sqrt{s}}I$$

Solving the above differential equation we get

$$\ln I(s) = -2k\sqrt{s} + \ln c \quad \Rightarrow \quad I(s) = ce^{-2k\sqrt{s}}$$

Further note that

$$I(0) = \int_0^\infty e^{-u^2}\mathrm{d}u = \frac{\sqrt{\pi}}{2} \quad \Rightarrow \quad c = \frac{\sqrt{\pi}}{2}$$

Therefore, we get

$$I(s) = \frac{\sqrt{\pi}}{2}e^{-2k\sqrt{s}}$$

Substituting this value in the equation (43.1), we obtain

$$L\left[\text{erf}\left(\frac{k}{\sqrt{t}}\right)\right] = \frac{2}{s\sqrt{\pi}}\left[\frac{\sqrt{\pi}}{2} - \frac{\sqrt{\pi}}{2}e^{-2k\sqrt{s}}\right] = \frac{1 - e^{-2k\sqrt{s}}}{s}$$

Taking inverse Laplace transform on both sides we get

$$\text{erf}\left(\frac{k}{\sqrt{t}}\right) = L^{-1}\left[\frac{1}{s}\right] - L^{-1}\left[\frac{e^{-2k\sqrt{s}}}{s}\right] = 1 - L^{-1}\left[\frac{e^{-2k\sqrt{s}}}{s}\right]$$

This leads to the desired result as

$$L^{-1}\left[\frac{e^{-2k\sqrt{s}}}{s}\right] = 1 - \text{erf}\left(\frac{k}{\sqrt{t}}\right) = \text{erf}_c\left(\frac{k}{\sqrt{t}}\right)$$

43.3 Dirac-Delta Function

Often in applications we study a physical system by putting in a short pulse and then seeing what the system does. The resulting behaviour is often called *impulse response*. Let us see what we mean by a pulse. The simplest kind of a pulse is a simple rectangular pulse defined by

$$\varphi_\epsilon^a(t) = \begin{cases} 0 & \text{if } t < a, \\ 1/\epsilon & \text{if } a \le t < a + \epsilon, \\ 0 & \text{if } a + \epsilon \le t. \end{cases}$$

Let us take the Laplace transform of a square pulse,

$$L\left[\varphi_\epsilon^a(t)\right] = \int_0^\infty e^{-st}\varphi_\epsilon(t)\,\mathrm{d}t$$

Substituting the value of the function we obatin

$$L\left[\varphi_\epsilon^a(t)\right] = \frac{1}{\epsilon}\int_a^{a+\epsilon} e^{-st}\,\mathrm{d}t$$

On integration we get

$$L\left[\varphi_\epsilon^a(t)\right] = \frac{e^{-sa}}{s\epsilon}\left[1-e^{-s\epsilon}\right]$$

We generally want ϵ to be very small. That is, we wish to have the pulse be very short and very tall. By letting ϵ go to zero we arrive at the concept of the *Dirac delta function*, $\delta(t-a)$. Thus, the Dirac-Delta can be thought as the limiting case of $\varphi_\epsilon(t)$ as $\epsilon \to 0$

$$\delta(t-a) = \lim_{\epsilon\to 0}\varphi_\epsilon^a(t)$$

So $\delta(t)$ is a ”function” with all its ”mass” at the single point $t=0$. In other words, the Dirac-delta function is defined as having the following properties:

(i) $\delta(t-a) = 0, \;\; \forall t, \; t \neq a$

(ii) for any interval $[c,d]$

$$\int_c^d \delta(t-a)\;dt = \begin{cases} 1 & \text{if the interval } [c,d] \text{ contains } a\text{, i.e. } c \le a \le d, \\ 0 & \text{otherwise.} \end{cases}$$

(iii) for any interval $[c,d]$

$$\int_c^d \delta(t-a)f(x)\;dt = \begin{cases} f(a) & \text{if the interval } [c,d] \text{ contains } a\text{, i.e. } c \le a \le d, \\ 0 & \text{otherwise.} \end{cases}$$

Unfortunately there is no such function in the classical sense. You could informally think that $\delta(t)$ is zero for $t \neq 0$ and somehow infinite at $t=0$.

As we can integrate $\delta(t)$, let us compute its Laplace transform.

$$L\left[\delta(t-a)\right] = \int_0^\infty e^{-st}\delta(t-a)\;dt = e^{-as}$$

In particular,

$$L\left[\delta(t)\right] = 1.$$

Remark: *Notice that the Laplace transform of $\delta(t-a)$ looks like the Laplace transform of the derivative of the Heaviside function $u(t-a)$, if we could differentiate the Heaviside function. First notice*

$$\mathcal{L}\big[u(t-a)\big] = \frac{e^{-as}}{s}.$$

To obtain what the Laplace transform of the derivative would be we multiply by s, to obtain e^{-as}, which is the Laplace transform of $\delta(t-a)$. We see the same thing using integration,

$$\int_0^t \delta(s-a)\, ds = u(t-a).$$

So in a certain sense

$$\text{"}\frac{d}{dt}\big[u(t-a)\big] = \delta(t-a)\text{"}$$

This line of reasoning allows us to talk about derivatives of functions with jump discontinuities. We can think of the derivative of the Heaviside function $u(t-a)$ as being somehow infinite at a, which is precisely our intuitive understanding of the delta function.

43.3.1 Example

Compute $L^{-1}\left[\frac{s+1}{s}\right]$.

Solution: We write,

$$L^{-1}\left[\frac{s+1}{s}\right] = L^{-1}\left[1+\frac{1}{s}\right] = L^{-1}[1] + L^{-1}\left[\frac{1}{s}\right] = \delta(t) + 1.$$

The resulting object is a generalized function which makes sense only when put under an integral.

Suggested Readings

Debnath, L. and Bhatta, D. (2007). Integral Transforms and Their Applications. Second Edition. Chapman and Hall/CRC (Taylor and Francis Group). New York.

Dyke, P.P.G. (2001). An Introduction to Laplace Transforms and Fourier Series. Springer-Verlag London Ltd.

Schiff, J.L. (1999). The Laplace Transform: Theory and Applications. Springer-Verlag, New York Inc.