Lesson 15

Tracing of Curves

15.1 Introduction

Now we use some mathematical techniques to trace curves and graphs of functions much more efficiently. We shall especially look for the following aspects of the curve.

1. Intersection with the coordinate axes.
2. Critical points
3. Regions of increase
4. Regions of decrease
5. Maxima and minima (including local ones)
6. Behaviour as x becomes large positive and large negative.
7. Values of x near which y becomes large positive or large negative.
8. Regions where the curve is convex up or down.
9. Asymptotes of the curve
10. Find whether the curve is symmetric

15.2 Behaviour as x becomes very Large

Suppose we have a function f defined for all sufficiently larger numbers. Then we get substantial information concerning our function by investigating how it behaves as x becomes large.

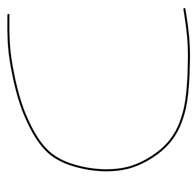
For example, $\sin x$ oscillates between -1 and +1 no matter how large x is.

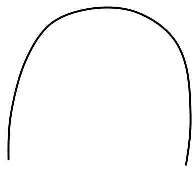
However, polynomials do not oscillate. When f(x) = x^2 as x becomes large positive. So does x^2. Similarly with the function x^3, or x^4 (etc.). We consider this systematically.

Example 15.1 Consider a parabola,

$$y = ax^2 + bx + c, \text{ with } a \neq 0.$$

There are two essential cases, when a > 0 or a < 0. We have the parabola which looks like in the figure





$y = ax^2 + bx + c$ $\qquad$ $y = ax^2 + bx + c$

$a > 0$ $\qquad$ $a < 0$

We look some numerical examples.

Example 15.2 Sketch the graph of the curve

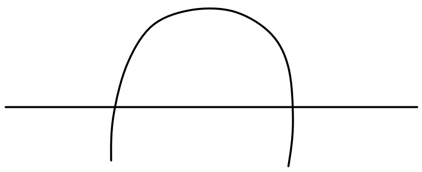
$$y = f(x) - 3x^2 + 5x – 1$$

We recognize this as a parabola.

$$f(x) = x^2(-3 + \frac{5}{x} - \frac{1}{x^2}),$$

when x is large positive or negative, then x^2 is large positive and the factor on

the right is close to -3. Hence f(x) is large negative. This means that the parabola has the shape as shown in figure.



We have $f'(x) = -6x + 5$. Thus $f'(x) = 0$ iff $x = \frac{5}{6}$. There is exactly one critical point. We have $f\left(\frac{5}{6}\right) = -3\left(\frac{5}{6}\right)^2 + \frac{25}{6} - 1 > 0$

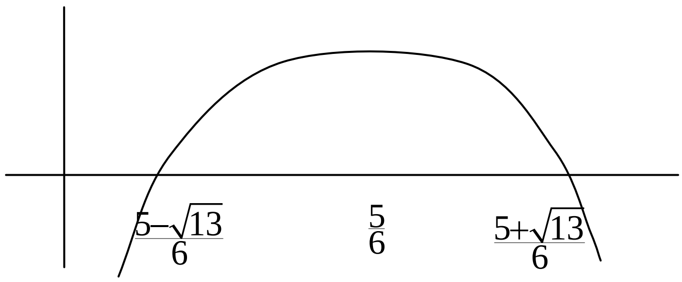
The critical point is a maximum, because we have already seen that the parabola bends down.

The curve crosses the x-axis exactly when

$$-3x^2 + 5x - 1 = 0$$

$$x = \frac{-5 \pm \sqrt{25-12}}{-6} = \frac{5 \pm \sqrt{13}}{6}$$

Hence the graph of the parabola looks as on the figure.



Bending down or convex upward

The same principle applies to sketching any parabola.

(i) Looking at what happens when x becomes large positive or negative tells us whether the parabola bends up or down.

(ii) A quadratic function

$$f(x) = ax^2 + bx + c \text{ with } a \neq 0$$

has only one critical point, when

$f'(x) = 2ax + b = 0$

So $x = \dfrac{-b}{2a}$

Knowing whether the parabola bends up or down tells us whether the critical point is maximum or minimum, and the value $x = \dfrac{-b}{2a}$ tells us exactly where this critical point lies.

(iii) The points where the parabola crosses the x-axis are determined by the quadratic formula.

Example 15.3. (Cubics) Consider a polynomial

$f(x) = x^3 + 2x - 1$, find $f(x)$ when $x \to \pm\infty$. We have

We can write it in the form

$$x^3\left(1 + \frac{2}{x^2} - \frac{1}{x^3}\right) \text{and, when } x \to +\infty \text{ means } f(x) \to +\infty$$

Example 15.4. (a) Consider the quotient polynomials like

$$Q(x) = \frac{x^3 + 2x - 1}{2x^3 - x + 1}$$

Here if $x \to \pm\infty$, then $Q(x) \to \dfrac{1}{2}$.

Example 15.4(b) Consider the quotient $Q(x) = \dfrac{x^3 - 1}{x^2 + 5}$

Here $\lim\limits_{x \to +\infty} Q(x) = +\infty$ *and* $\lim\limits_{x \to -\infty} Q(x) = -\infty$

The meaning of the above limit is that there is no number which is the limit of Q(x) as $x \to +\infty$ or $x \to -\infty$.

We can now sketch the graphs of cubic polynomials symmetrically.

Example 15.5 Sketch the graph of f(x) = x^3 – 2x + 1

1. If $x \to +\infty$ then $f(x) \to +\infty$

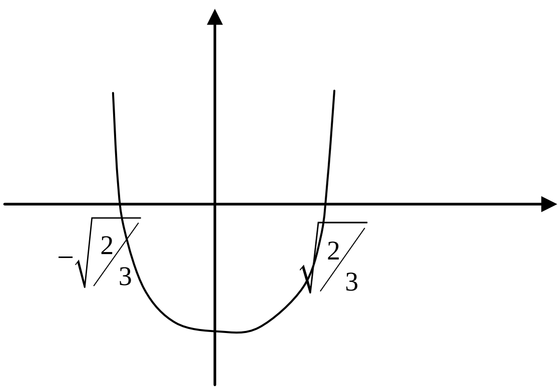
   If $x \to -\infty$ then $f(x) \to -\infty$

2. We have $f'(x) = 3x^2 - 2$

$$f'(x) = 0 \Leftrightarrow x = \pm\sqrt{2/3}$$

The critical points of f are $x = +\sqrt{2/3}$ and $x = -\sqrt{2/3}$.

3. Let $g(x) = f'(x) = 3x^2 - 2$. Then the graph of g is a parabola which is given as



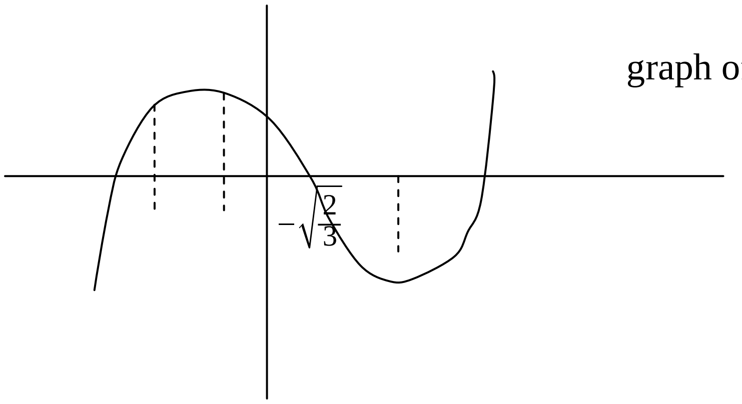
Graph of $g(x) = f'(x)$

Therefore, $f'(x) > 0 \Leftrightarrow x > \sqrt{2/3}$ *and* $x < -\sqrt{2/3}$, where g(x) > 0 and f is strictly increasing on the intervals $x \geq \sqrt{2/3}$ and $x \leq -\sqrt{2/3}$.

Similarly $f'(x) < 0 \Leftrightarrow -\sqrt{2/3} < x < \sqrt{2/3}$ where g(x) < 0, and f is strictly decreasing on this interval . Therefore $-\sqrt{2/3}$ is a local maximum for *f*, and $\sqrt{2/3}$ is a local maximum.

4. $f''(x) = 6x$, and $f''(x) > 0$ iff $x > 0$ and $f''(x) < 0$ iff for *x* > 0, therefore *f* is bending up (convex downward) for *x* > 0 and bending down (convex upward) for *x* < 0. There is an inflection point at *x* = 0.

Putting all this together, we find that the graph of f looks like this



graph of $f(x) = x^3 - 2x + 1$

Example 15.6 Sketch the graph of the curve.

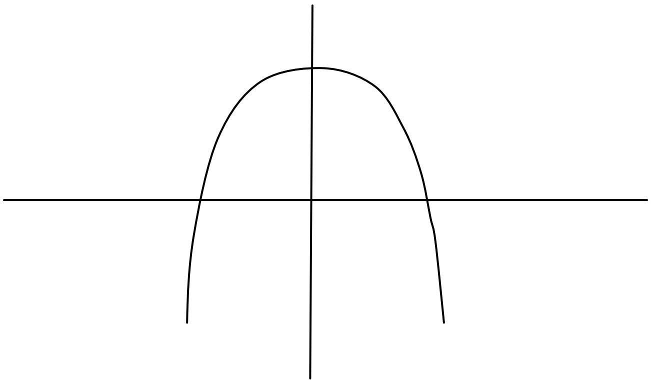
y = - x^3 + 3x - 5

1. When x = 0 , we have y = -5. With general polynomial for degree ≥ 3 there is in general no simple formula for those x such that f(x) = 0, so we do not give explicitly in the intersection of the graph with the *x* – axis.

2. The derivative is $f'(x) = -3x^2 + 3$

$$f'(x) = 0 \Leftrightarrow x = \pm 1$$

The graph of $f'(x)$ is given by



$f''(x) = -6x$, $f''(1) = -6$, $f''(-1) = 6$, $f''(x) > 0$ iff x < 0 and $f''(x) < 0$ iff x > 0. x = 0 is an inflection point x = 0.

f is strictly decreasing $\Leftrightarrow$ $f'(x) < 0$

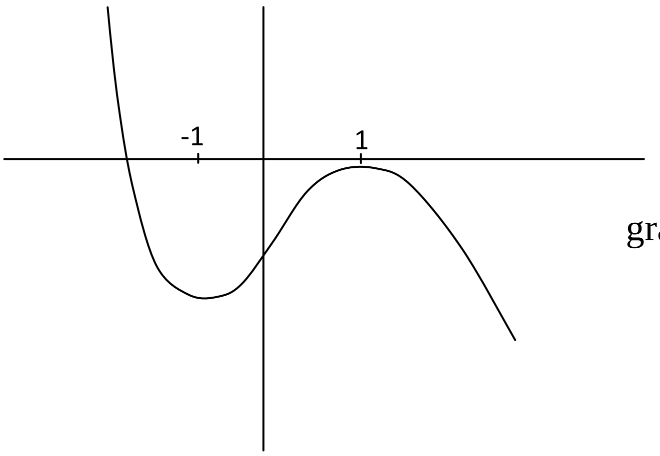
$$\Leftrightarrow \ x < -1 \ \textit{and} \ \ x > 1$$

f is strictly increasing $\Leftrightarrow$ $f'(x) > 0$

$$\Leftrightarrow \ -1 < x < 1 \ .$$

Therefore f has a local minimum at x = -1 and local maximum at x = 1.

Putting all this information together, we see that graph of f looks like this



graph of $f(x) = - x^3 + 3x + 5$

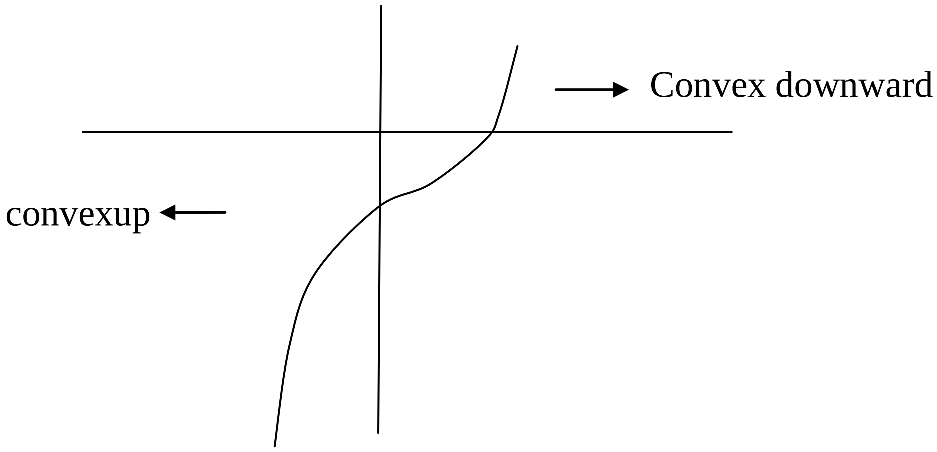
Example 15.7 Let $f(x) = 4x^3+2$. Sketch the graph of f.

Solution:

Here we have $f'(x) = 12x^2 > 0 \ \forall \ x \neq 0$. There is only one critical point, when x = 0. Hence the function is strictly increasing for all x, and its graph looks like

$f''(x) = 24x > 0 \ \ for \ all \ \ x > 0$

$f''(x) < 0 \ \ for \ \ x < 0$



Example 15.8 Sketch the graph of f(x) = $4x^3 + 4x$.

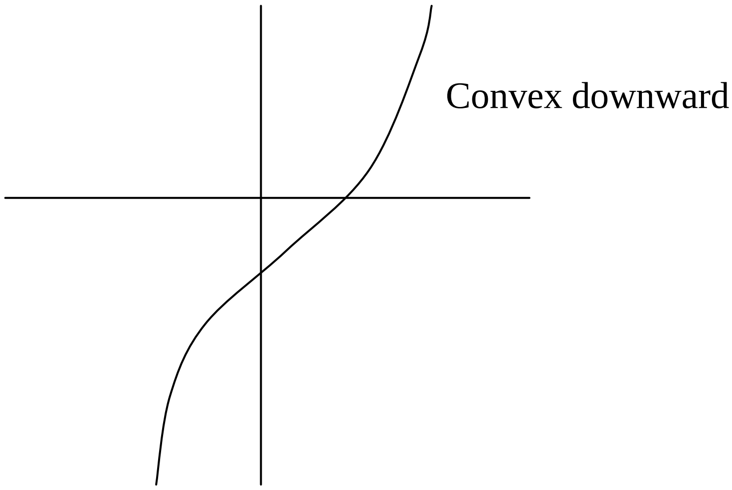
Solution:

$f'(x) = 3x^2 + 4 > 0 \ \forall \ x$

$f''(x) = 6x > 0 \ \ for \ \ x > 0$

$f''(x) < 0 \ \ for \ \ x < 0$

So the graph looks like



Convex upward

In both the above examples x = 0 is an inflection point.

15.3 Rational Functions

We shall now consider quotient of polynomials.

Example 15.9 Sketch the graph of the curve

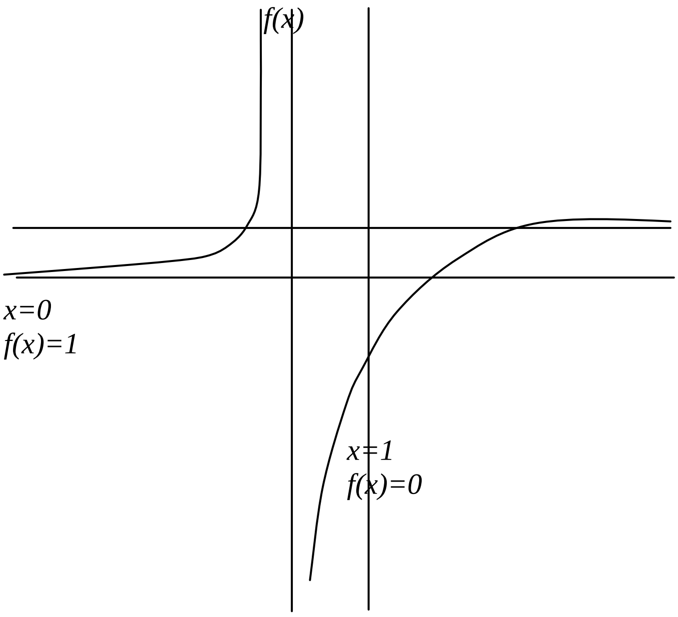
$$y = f(x) = \frac{x-1}{x+1}$$

1. When x = 0, we have f(x) = 1. When x = 1, f(x) = 0.

2. The derivative is $f'(x) = \frac{2}{(x+1)^2}$

It is never zero, so the function has no critical points.

3. The denominator is a square and hence is always positive, whenever it is defined, i.e., for $x \neq -1$. Thus $f'(x) > 0$ for $x \neq -1$. The function is not defined at x = -1 and hence derivative also is not defined at x = -1, i.e., f(x) is increasing in the region x < -1 and is increasing in the region x > -1



4. There is no region of decreasing.

5. Since the derivative is never zero, there is no relative maximum or minimum.

6. The second derivative is $f''(x) = \dfrac{-4}{(x+1)^3}$.

There is no inflection point since $f''(x) \neq 0$ for all x where the function is defined. If x < -1, $(x + 1)^3 < 0$, and $f''(x) > 0$, f(x) is bending up or convex downward. If x > -1, then x+1 > 0 $\Rightarrow$ $(x+1)^3 > 0$. So $f''(x) < 0$ i.e., *f(x)* is bending down (convex upward).

7. As $x \to \infty$, f(x) $\to 1$ $\quad f(x) = \dfrac{x-1}{x+1} = \lim\limits_{x\to\infty} \dfrac{x\left(1-\dfrac{1}{x}\right)}{x\left(1+\dfrac{1}{x}\right)} = 1$

when $x \to -\infty$, f(x) $\to 1$

8. As $x \to -1$, the denominator approaches 0 and the numerator approaches -2. If x approaches -1 from the right so x > -1, then the denominator is +ve and the numerator is negative. Hence the function $\frac{x-1}{x+1}$ is negative , and is large negative. Putting all these information we get the graph looks like the given figure.

EXERCISES

Sketch the following curves, indicating all the information stated in the examples etc.

1. $y = \frac{x^2+2}{x-3}$

2. $y = \frac{x-3}{x^2+1}$

3. $y = x^4 + 4x$

4. $y = x^8 + x$

5. $f(x) = x^4 + 3x^3 - x^2 + 5$

6. $y = \frac{x^2-1}{x}$

7. Show that a curve $y = ax^3 + bx^2 + cx + d$ with $a \neq 0$ has exactly one inflection point.

Keywords: Curve tracing, increasing, decreasing, convex up, convex down.

References

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Jain, R. K. and Iyengar, SRK. (2010). Advanced Engineering Mathematics, 3 rd Edition Publishers, Narsa, India.

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Suggested Readings

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